

# **EXPLORATIONS IN MATHEMATICS**



# EXPLORATIONS



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ROBERT B. DAVIS

*Professor of Mathematics and Education  
Syracuse University and Webster College*

# IN MATHEMATICS

## A TEXT FOR TEACHERS

Containing topics, techniques, methods, and materials  
for a supplementary course in introductory  
mathematics and including  
the *Student Discussion Guide*, with answers

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## PREFACE

The experimentation which has been carried on since 1957 by the Madison Project owes many debts to many people. I could not even determine whose contributions have been the largest. At this moment I can think of more names to list than I can possibly find space for. To those friends and colleagues who are not mentioned here, I want to say that you are not forgotten.

Among those whose contributions have been very great indeed are Professor Donald E. Kibbey (Chairman of the Mathematics Department at Syracuse University), Dean Lawrence Schmeckebier, Mrs. Jane Downing, Mrs. Doris McLennan, Miss Marie Lutz, Miss Cynthia Parsons, and Mr. William Bowin, who helped tremendously in the early days of the Project when it operated mainly in the area of Syracuse, New York. The subsequent sophisticated experimental work done in Weston, Connecticut, was possible only because of the inspired efforts of Mrs. Beryl S. Cochran, greatly assisted by Gilbert Brown and Herbert Barrett of the Weston, Connecticut, public schools. The recording of this work via motion picture films and audio-tape was made possible by Morton Schindel, President of Weston Woods Studios. The later expansion of the Project's efforts has depended heavily upon the assistance of the retiring President of Webster College, Sister M. Francetta Barberis, S. L.; her successor, Sister M. Jacqueline Grennan, S. L.; the Chairman of the Webster College Mathematics Department, Professor Katharine Kharas; J. Robert Cleary of Educational Testing Service; Sister Francine, S. L., of Nerinx High School; Frank H. Duval, of McKnight Elementary School, University City, Missouri; and Ruth Hertlein and Gerald Baughman of Hilltop Elementary School, Ladue, Missouri.

Madison Project materials have been developed initially in suburban schools. The adaptation of this material, or parts of it, for use with culturally deprived children in large cities has been made possible by the administrative leadership of Dr. Samuel Shepard in St. Louis and Dr. Evelyn Carlson in Chicago, assisted by Ogie Wilkerson, Bernice Antoine, Emma Lewis, Gail Saliterman, and others, and by Dr. John Huffman in San Diego County, California.

Mathematical and pedagogical ideas in this volume have been contributed by many people, including Professors Robert Exner, Erik Hemmingsen, and Thomas Clayton, of Syracuse University; Professors Andrew Gleason and Frederick Mosteller of Harvard University; Professor Gerald Thompson of the Carnegie Institute of Technology; H. Stewart Moredock of Sacramento State College; Dr. William Reddy of the U.S. Army Ordnance Corps; and Donald Cohen and Knowles Dougherty of the Madison Project.

All modern efforts at curriculum revision owe a profound debt to several dominant national and world leaders—such as Max Beberman, David Page, Warwick Sawyer, Leonard Sealey, and Caleb Gattegno—and especially to the men who may be said to have created the present era: Jerrold Zacharias and Francis Friedman of the Massachusetts Institute of Technology; David Hawkins of the University of Colorado; Phillip Morrison of Cornell University; Jerome Bruner of Harvard; the late Richard Paulson, John Mays, Senta Raizen, and Charles Whitmer of the National Science Foundation; and their colleagues. Considering the consequences of their activities, the innovators themselves are incredibly few in number—but their work is being taken up by many others, and it is now realistic to hope that a new approach to the curriculum revision will soon exist in the United States.

Preliminary trials of *Explorations in Mathematics* were conducted by Gordon Clem at St. Thomas Choir School in New York City, by Elizabeth Herbert and Lyn McLane at Weston, Connecticut, and by others in various schools and places.

The myriad tasks of assembling a printable manuscript were supervised by Mrs. Bernice Talamante in my office and by the editors and staff at Addison-Wesley.

All of the teachers and administrators who have worked with the Project have chosen to spend much of their lives studying the culture which history passes on to our generation, selecting (as much as we are free to do so) that portion which we should pass on to our children, and finding suitable ways for doing this. It is not an unworthy task. Any value in this book is the fruit of their labor.

Robert B. Davis

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## THE PROJECT

### *A supplementary program*

This book is intended to provide children with various "creative learning experiences" in mathematics. These experiences are appropriate for a wide range of ages and grade levels: some can be used with second graders (i.e., chronological age about 7 years), while all can be used with first-year high school students, provided they have not previously learned this material. In general, the materials have been assembled with students in grades 5 through 9 in mind.

The material presented in this book was developed as part of extensive experimental teaching conducted by the Syracuse University—Webster College Madison Project,\* under the direction of Professor Robert B. Davis and Mrs. Beryl S. Cochran.

Because many aspects of these discovery lessons may strike some readers as novel and unusual and because this book does not stand alone—but is part of a large array of Madison Project materials presented via books, pamphlets, articles, tape recordings, and sound motion picture films—some explanation is required.

\*The name of the Project is taken from the name of the Madison School, in Syracuse, N. Y., where the Project's earliest experiments were conducted.

This book is intended to help teachers provide a supplementary program in "modern" mathematics. Briefly, features of this program include the following:

(1) These materials supplement, but do not replace, the usual school program in arithmetic and science.

(2) Use of Madison Project materials can (and generally should) be introduced into a school system gradually; after a first step-by-step introduction, use of the materials can be expanded and extended, allowing progressive revision and growth within the school program. Project materials are being prepared and augmented continually, allowing for unlimited "open-ended" growth in the school mathematics program.

(3) These supplementary, "modern" mathematics materials may be taught in one, two, or three lessons per week, perhaps for a total time of one hour per week. Alternatively, portions of this book may be used as "units," occupying an occasional week or two during the school year. For grades 7 and 8, this latter arrangement is probably preferable.

(4) As mentioned previously, some of the materials in this book may be started as early as grade 2; some may be used as late as senior high school. Considerable variation is possible to meet the needs of individual classes and individual school situations.

(5) The mathematical content combines certain fundamental ideas which underlie nearly all mathematics (such as variable, open sentence, number line, Cartesian coordinates, truth set, function, etc.), together with some important "new" topics that are basic to modern uses of mathematics (such as matrices, logic, statistics, etc.).

## 2 THE PROJECT

(6) Emphasis is placed on creative, informal exploration by the children; rote drill is avoided entirely, as not appropriate to the objectives of a "modern" mathematics program.

(7) Extensive teacher training in the use of these materials is available from the Madison Project, Webster College, St. Louis, Missouri 63119, or from the Madison Project, Syracuse University, Syracuse, New York 13210.

(8) Films showing actual classroom lessons and other materials useful to teachers, administrators, and parents are available from the Madison Project at either of the college addresses listed above (see also Appendix B).

(9) This book has been arranged so that it may be used independently of the companion volume, *Discovery in Mathematics* (Addison-Wesley, 1964). (We shall hereafter refer to these two books as *Explorations* and *Discovery*.) If both books are used in sequence, it is probably better to use *Discovery* first, and to follow it with the appropriate sections of *Explorations*. It is probably better still to use the two books simultaneously, selecting sections to suit your class.

### *Why it is needed*

Having taught these materials for some seven years prior to assembling them in this book, we naturally have many ideas and suggestions. We shall make only one or two at this time.

#### **Broadening the school program**

Obviously, something is happening to the school curriculum these days. Mathematics is not alone. In nearly every area we seem to be realizing that the school program has been confined within arbitrary, narrow limits that are not appropriate in today's world.

Consider history and geography. Traditionally, students in the United States have studied "Western" culture—France, Germany, Italy, England, and the United States itself, for the most part—and have learned little or nothing about Asia and Africa. Yet what is the world in which our citizens live? The main crises of recent years have involved the Congo, South Africa, Vietnam, Korea, Cyprus, Algeria, Ghana, Cuba, Formosa, . . . , areas about which most of us know little. It seems clear that our study of human culture must be broadened to include more than merely "Western" culture.

In the case of music, the traditional school program usually gave the impression that music began with Bach and ended with Brahms—or possibly Sibelius. Music educators

are now at work trying to broaden this far too narrow slice, and to include pre-Bach music, ethnic music, jazz, folk songs, and contemporary concert music.

One can look to nearly any area in the school program and observe a similar broadening taking place. This broadening promises to produce a curriculum more vital, more relevant, more honest, and more useful than could be achieved within the arbitrary and narrow confines of the past.

In the case of mathematics, the traditional K-8 curriculum was concerned mainly with adding, subtracting, multiplying, and dividing, and with applications to simple retail business transactions. The 9-12 program was dominated by the mathematics related to surveying, navigating, manual computation, and similar tasks.

Today's uses of mathematics are far broader, and a wider slice of mathematics needs to be presented in our schools. This broadening of the school mathematics program is—as we shall see in the following pages—one of the main purposes of this book.

#### **Creativity**

That children are *creative* is hardly a new idea, nor is the notion that schools can properly nurture this creativity. Unfortunately, however, creativity has known its own bounds. It has been recognized in relation to "creative writing," to the composing of music, the writing of poetry, the production of plays. It is an acknowledged ingredient in painting, sculpture, and ceramics. But . . . *creativity in mathematics*? Hardly! Mathematics has been usually regarded as a matter of flash cards for addition facts, and doing what you were told to do in a computation with logarithms. Mathematics has seemed to be a matter of drill and following directions. Mathematics has simply born no relationship whatsoever to the task of nurturing *creativity*.

Yet everyone who has studied mathematics at any depth knows how wrong this picture is. The *routine* aspects of mathematics have never been the more rewarding or valuable part. Today, when electronic computers are taking over all of our routine tasks, the old view of mathematics is even more seriously in error. Nowadays, routine mathematics is a task for machines; only creative mathematics is a proper task for humans.

This book makes every possible effort to see that the child's experiences with mathematics shall be creative and not routine.

Suggestions for further reading are given in Appendix A, under the heading "Philosophy and Pedagogy." See particularly: Davis (23), (24), (25), (26), (27); Goals for School Mathematics (63); Mearns (4); Schwab (1); and Torrance (70).





## THE APPROACH

### *Creative learning experiences*

Although most educators and scientists who have either viewed Madison Project films or observed Madison Project lessons have been markedly enthusiastic, there have been some few who have been puzzled and disappointed. These latter have often said, "There was no teaching in that lesson." Perhaps not – but there was a great deal of *learning*.

We believe that those who see "no teaching" in Madison Project lessons are disappointed at not recognizing what they regard as the essential structure of a lesson: the teacher first telling the students what will happen, then showing them what to do and how to do it, then giving them practice or drill, and, finally, summarizing the lesson.

To be sure, none of these "parts of a lesson" can be observed in typical Madison Project lessons. Their absence is deliberate and important.

Perhaps, then, in the eyes of some teachers, we do not present "lessons." What we do instead is to suggest to

the children one or more mathematical tasks, and then work with them, unobtrusively, as they devise their own methods for tackling the tasks. Seven years of Madison Project experience have convinced us that children can learn a great deal in this way. Some semantic clarification may, however, be achieved if we do not refer to these as "lessons"; we have instead introduced the phrase *creative learning experiences* (or, alternatively, *informal exploratory experiences*) to describe the "lessons" – or, if you prefer, the "nonlessons" – that are characteristic of Madison Project teaching.

Over the past few years, Madison Project films and live classes have been observed by a variety of professionals: teachers, mathematicians, motivational psychologists, clinical psychologists, logicians, physicists, guidance counselors, school principals, psychiatrists, psychoanalysts, cultural anthropologists, and linguists, among others. From these observers have come descriptive statements that shed considerable light upon what is, and what is not, a "creative learning experience" in the Madison Project sense.

**Concern for Basic Mathematics.** The Project's creative learning experiences are concerned with fundamental mathematical concepts, such as variable, open sentence, signed number, the number line, Cartesian coordinates, function, etc. They are *not* concerned with highly optional or artificial topics. Consequently, a sequence of creative learning experiences should add up to significant power in broad areas of mathematics.

**An Active Role for the Student.** As far as possible the student is given an active role to play. Passive roles, such as listening to a lecture or reading exposition, are usually avoided. The "active role," however, may refer to *mental activity*, as well as to *physical activity*. The child who leaves class with a look of puzzled involvement is playing an active role, quite as much as the student who is making a measurement with a meter stick. Even in listening, the best students have always played an active role, in terms of critical thinking, seeking alternatives, etc. What is sought here, in the words of David Page, is to get every student thinking the way the best students have always thought.

**Concepts Learned in Context.** We try to have the students learn concepts in context. Every mathematical concept or technique was developed to aid in attacking some kind of problem. When we tear the concept out of this context and attempt to state it *in vacuo* (as is too often the case), we render the concept unintelligible. Thus, in our informal exploratory experiences, we begin not with definitions but with *tasks*. The concepts unfold naturally as one seeks insight into the nature of the tasks.

**Opportunities for Discovery.** In every lesson we try to have opportunities for discovery lurking just beneath the surface. These discoveries are sometimes an essential part of the lesson, but often they have the effect of *going beyond* the basic lesson. In this latter case, it does not matter how many, or how few, students actually make the discovery.

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The point here is to get the children in the habit of "looking for patterns" whenever they are working in science or mathematics; the discovery—often accidental—of such patterns is, after all, the main device by which science moves forward.

As an example of one such lesson, consider the film "First Lesson."\* In this, we have children trying to put the same number in every  $\square$  in the open sentence

$$(\square \times \square) - (5 \times \square) + 6 = 0,$$

so as to obtain a true statement. Sooner or later they find that  $2 \rightarrow \square$  yields a true statement:

$$\begin{array}{rccccccc} (\square \times \square) & - & (5 \times \square) & + & 6 & = & 0 \\ (2 \times 2) & - & (5 \times 2) & + & 6 & = & 0 \\ 4 & - & 10 & + & 6 & = & 0 & \text{True} \end{array}$$

Similarly,  $3 \rightarrow \square$  yields a true statement:

$$\begin{array}{rccccccc} (\square \times \square) & - & (5 \times \square) & + & 6 & = & 0 \\ (3 \times 3) & - & (5 \times 3) & + & 6 & = & 0 \\ 9 & - & 15 & + & 6 & = & 0 & \text{True} \end{array}$$

Moreover, if you take any number other than 2 or 3 and put it in each box, the resulting statement will be false. We express all this by saying, "The open sentence

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

has the truth set

$$\{2, 3\}."$$

Now, the *basic* purpose of this lesson is to give the children experience with "variables" (i.e., with "putting numbers into the  $\square$ ") and to give them experience with "signed numbers" (such as  $9 - 15 = -6$ , etc.). There is, of course, an intriguing and important discovery for any child who sees it—but for the child who does *not* see it, it is not essential to the basic purpose of the lesson.

**Appropriateness to the Age of the Child.** We try to select informal exploratory experiences so that they are appropriate to the age of the child. Our experience thus far is not extensive enough to encompass all types of school situations or all varieties of cultural background; nonetheless, as a broad generality, we are finding fifth-graders (chronological age about 10 years) to be the "natural intellectuals," interested in science and abstract mathematics. Probably the intellectual curiosity of even younger children—even two- and three-year-olds—is also very great, but their command of abstract symbolism is limited. At about grade 6, we usually encounter the beginning of a decline, which extends over grades 7 and 8 (chronological ages 12 and 13); during this time abstract mathematics seems often to lose some of its appeal for the child—he is more easily captivated by "engineering" or "activity" type

\*This lesson is also presented in Chapter 3 of *Discovery*. We put this example in at this point for illustration only. Please do not worry if any of it seems strange and confusing—we shall develop all of these ideas carefully in the pages which follow.

subjects, such as logic, statistics, and the physics of forces and velocities.

Our present generalities cannot be trusted too far, since we suspect that variations in school and cultural situations produce variations in student preferences. In general, we must leave it up to you, who actually know your own students, to select topics appropriate to their interests.

**Informality.** Good creative learning experiences usually seem to occur in a relatively informal atmosphere. This fact was first emphasized to us by a linguist\* who counted evidences of levels of formality within the spoken language (or gestures) of the teacher!

**Low Anxiety Level.** Good "nonlessons" of the type we are describing seem to be characterized by a low anxiety level. Every child may be eagerly participating, but there is relatively little fear of failure.

**Nonauthoritarian Nature.** We try to avoid an authoritarian atmosphere. So important is this that we try always to provide *autonomous decision procedures*—that is, procedures whereby a student can distinguish true statements from false statements, *without recourse to the teacher or to books*. For the very young child, the process of counting often serves as an autonomous decision procedure. Is it true that  $3 + 4 = 7$ ? If the child can count reliably, he can count out three objects, count out four objects, combine them, and count the result.

For the mature mathematician, some combination of intuition and logic ostensibly provides an autonomous decision procedure, but history records many instances where the separation of "true" from "false" has been difficult and uncertain. At every age level, we try to provide an autonomous decision procedure if we can possibly do so. "Postman stories" do this for the arithmetic of signed numbers. Substituting coordinates into the open sentence often does this for conjectures about graphs. In many cases, the ability to solve the problem by a variety of different methods, thereafter comparing results, provides an autonomous decision procedure. In all such cases, the child can settle for *himself* the truth or falsity of the statement in question.

**The "Light Touch."** We use a "spiral" approach. A subject is not pursued too heavily within a single session, but recurs from time to time, and in various guises, until it becomes familiar.

**Intrinsic Motivation.** One psychologist who advises the Project gave a lecture in a college seminar room where tea was served. At the start of the lecture, he balanced a half-full cup (of the college's best china) on the edge of the table, just at the point where it teetered back and forth and threatened at any moment to fall to the floor (on the college's best carpet). While the cup teetered, the psychologist lectured.

No one heeded his words; all eyes were on the cup. Probably everyone in the room felt acutely uncomfortable, and

\*Professor H. A. Gleason, Jr., of Hartford Seminary.

wanted to walk over and push the cup further onto the table. Finally, one person did.

Why did everyone feel a need to perform this task? Not for any *extrinsic* reason—it was not *our* china, nor *our* rug, nor would we have to clean up, nor would we be blamed. We were not *paid* to push the cup onto the table—we were not even asked to do so! Yet everyone *wanted* to, and wanted to very much.

This is an instance of what we have called *intrinsic* motivation. The task itself cries out to be done. Other examples might be finding a key word in a crossword puzzle or finding a long-sought piece in a jigsaw puzzle.

We try, wherever possible, to see that the mathematical tasks that we suggest to the children possess this peculiar compelling nature: you feel that you *want* to do it. We make very little use of extrinsic rewards—indeed, some research appears to indicate that extrinsic rewards can stand in the way of genuine creativity.

For further reading, see Appendix A: Arons (100); Cantor (13); Davis (28); Gage (115); Holt (3), (123); Kelly (41); Sawyer (65); Schwab (1); Skinner (66); and Torrance (70).

## Objectives for student growth

No important human activity is strictly bound by its apparent objectives; on the contrary, it goes beyond these objectives and may end up possessing values hardly contemplated at the outset. We would like to think that virtually all educational activity has this definition-defying character, and that Madison Project teaching is no exception. It may, however, be useful to consider a brief list of "objectives" of Madison Project teaching. These objectives refer to objectives for the growth, over the years, of an individual student. (The Project believes that the teacher should also continue to learn and to grow and that the school program should continue each year to grow and to improve. The present list of objectives refers, however, merely to the growth that we should like to observe in each individual student.) The list is surely incomplete, but it may prove suggestive.

(1) We want children to develop their ability to discover patterns in abstract situations.

(2) We want children to develop the kind of independent exploratory behavior that goes beyond anything the teacher suggested, that explores paths that both teacher and textbook author have overlooked, that sees open-ended possibilities for extension where others would see only closed completion of the assigned task. (And we do not want the children "going on" merely in order to please *us*; we *do* want them exploring beyond the boundaries of the day's lesson *because they feel there are some intrinsically rewarding things outside those boundaries!* One might say that this



is the difference between the dog who escapes from a fenced-in yard because he wants to explore what's outside of the fence versus the dog who heels because he has been trained to heel.)

(3) We want the children to acquire a set of mental symbols which they can manipulate in order to "try out" mathematical ideas. (This point will be discussed more fully when we come to study the arithmetic of signed numbers.) Probably all good mathematicians possess such a set of mental symbols, although they may be unable to describe them in words.

(4) We want the children to learn the really fundamental mathematical ideas, such as variable, function, graph, matrix, isomorphism, and so on, and we want these ideas to be learned early enough in life so that they can serve as the foundation on which to build subsequent learnings.

(5) We want the children to acquire a reasonable degree of mastery of important techniques.

(6) We want them to know basic mathematical facts—for example, the fact that  $1 \times 1 = 1$ .

The objectives listed above are rather specific, mathematical objectives that might be described as "cognitive." There are also other important objectives, of a more general nature.

(7) We want our students to emerge from our classes with a genuine belief that *mathematics is discoverable*.

(8) We want them to be able to make a realistic assessment of their own individual ability to discover mathematics. (For nearly all students, this ability is greater than they initially realize.)

(9) We hope they will genuinely recognize the openness of mathematics.

(10) We hope they will develop an honest self-critical ability. This is important in mathematics, as in nearly everything else. It is no virtue to defend an incorrect line of reasoning, nor does habitual defensive action facilitate further learning.

(11) We hope our students acquire a personal commitment to the value of abstract rational analysis.

(12) We hope the students will come to value "educated intuition." The shrewd guess is never to be despised!

(13) We hope our students will come to feel that mathematics is "fun" or "exciting" or "challenging" or "rewarding" or "worthwhile."

(14) We want our students to learn something of the culture that lies behind twentieth-century man. We want them to understand mathematical history *because they have lived through it*. We can bring history right into the classroom: the students can live through experiences such as trying to solve  $x^2 = 4$ , only to find their path blocked, until finally someone makes a brilliant suggestion and they are able to move ahead. They have just witnessed a significant historical breakthrough, and they can consequently understand what this means in the history of mathematics in general. Because they have seen mathematics discovered, beheld this with their own eyes and heard it with their own ears, they can *understand* the process by which mathematics in general is discovered.

(15) Finally, we want our students to be able to appreciate pure mathematics for its own sake, but at the same time to be able to see mathematics in a natural relation to physics, biology, and so on.

## ***Experience, intuition, and explicit formulations***

One of the guiding principles in the sequencing of the Madison Project materials is that we should avoid asking children to discuss things they have never done. Instead, we advocate a sequence wherein the child first gets experience, then (as a result) develops intuitive ideas, and finally strives for explicit words and symbols to describe his experience. This may seem obvious, but mathematics teaching very frequently violates this sequence.

For example, it is common to *start* with one or more *definitions*. Although this may at first glance seem logical, a more careful look usually shows it to be unsatisfactory. A *definition* is more nearly *the place to end up*, rather than the place to start. When we state a definition, we are surveying an intellectual landscape and deciding which ideas are "in" and which are "out." Such a setting of precise boundaries cannot reasonably come until we have some rough general familiarity with the landscape.



Our recommended sequence somewhat resembles the way man is exploring the moon. First we have the very broad and rough ideas—for example, that the moon is there. Then we get a slightly finer, but still quite rough, version—the appearance of the moon's surface as seen through a telescope on earth. Then we get still more detailed versions—the photos sent back by space vehicles which crashed into the moon, or passed near it. We shall soon be getting yet finer detail, from space vehicles that achieve a "soft" landing on the moon.

Thus we get a sequence: very rough ideas, rough ideas, moderately refined ideas, more minutely detailed ideas, etc. One consequence of this is that we often allow "errors" to pass unchallenged in the early stages of a discussion, *because we feel that the time has not yet come to raise such an issue*. Later, when the child's experience is more extensive, his intuition more fully developed, and his ability to discuss his intuitive ideas in explicit language is greater, *then* we can discuss matters of finer detail.

Notice that this means that we do not "try to get things exactly right from the very beginning." We try to see that the *teacher* does not mislead the child unnecessarily, but we do not expect sophisticated accuracy in the child's answers or suggestions. The finer detail comes later. For example, if a child says that the open sentence

$$a \times \square = b$$

has the truth set  $\{b/a\}$ , we would not necessarily remind him of the case where 0 is substituted for  $a$ . We would *if we felt sure that he was ready for this*. Otherwise we would let this matter pass unnoticed for the moment. Another way to say this is: *don't answer a question until after it has been raised*. (Of course, if you feel the students are ready, you can see that the question is raised.)

Professor Morris Kline says, "Sufficient unto the day is the logic thereof." One might add, "Sufficient unto the day is the level of sophistication thereof." And so on . . .

Premature consideration of exceptional cases, complications, and other forms of pathology is *not* beneficial to most students. They do not have the experience, the intuition, or the explicit language to permit them to make use of such considerations. This must be developed gradually, and in the sequence: experience, then intuition, then abstract symbolism, vocabulary, and notation.

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# THE TOPICS

## *What should we teach?*

In the midst of today's remarkable educational flux it is hardly surprising that one often hears the question, *What should we teach?* Like many important questions, this probably admits of no absolute and definitive answer. Nonetheless, all of us who are involved with education are involved also with the question, *What should we be teaching?* Those of us working in the Madison Project experimentation are no exception, and, one way or another, this question is never far from our thoughts. Here, briefly, are some of our ideas on this subject:

In the first place, for an educational experience to be one which we *should* give to a child, we believe it should meet most or *all* of the criteria mentioned in the preceding pages: it should be appropriate to the age of the child, if possible it should be creative rather than routine (never fear for the routine tasks—somehow they always seem to be with us, and we need save no special niche to protect them!), the learner should play as active a role as possible, the motivation should be intrinsic rather than extrinsic, and so on. But there is another criterion of very great importance: *what the child knows today should provide the best possible basis for what he will wish to learn tomorrow.* However, achieving this is never so simple as merely saying it. Considerable insight into how this works can be gained from the psychological theory developed by Lewin, Piaget, Tolman, and Bruner.

## *The psychology of Lewin, Tolman, Piaget, and Bruner*

"Big things" can be made up from "little things" in a variety of ways. A journey of a thousand miles, say the Chinese, begins with but a single step. However, if we imagine building a long walk from a sequence of individual steps, taken alternately with the left and right feet, the relation of the whole to these parts is, for most purposes, not very illuminating. Now one can imagine building knowledge from a concatenation of small bits of information—left, right, left, right, as it were—but, as with the journey, the relation of part to whole is dismal and unilluminating. Felix Mendelsohn, a brilliant technician in matters of musical composition, is said to have written a symphony one measure at a time—a technical tour de force of great proportions. Such is not the usual method of composing music. The common, and far easier, method is to work out small pieces that have definite structural roles to play: a first theme, other themes, harmonic sequences, variations of the themes, contrapuntal themes, and so on. These parts relate to the whole in a way that the mind can grasp and manipulate.

Suppose we were learning our way around a strange city. The "putting one foot in front of the other" kind of synthesis might be attempted by taking a map, starting in the lower left-hand corner, and gradually learning the map by contiguous expansion—that is, "strip by strip," or something of that sort. This is the dreary, dull, and dismal ap-



8 THE TOPICS

proach, but it is nonetheless sometimes used in programmed instruction and in curriculum planning. Obviously, a far pleasanter approach—and a far more powerful one—is to seek here, also, for elements of structural significance. For the city of Syracuse, for example, one might begin by learning its “main street,” which is named Salina Street and runs (roughly) north and south, together with its principal east-west street, which is probably Genesee Street. These two elements produce a map which looks more or less like that in Fig. 1.

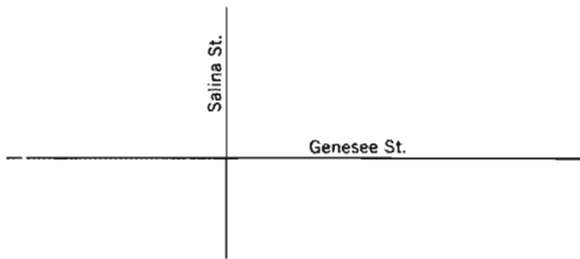


FIGURE 1

The psychologists Jean Piaget, Kurt Lewin, E. C. Tolman, and Jerome Bruner have presented an extended theoretical framework for discussing the cognitive structures that we all have “inside our minds,” and the process by which these cognitive structures become modified or yield their place to newer, more up-to-date versions. If we take the preceding two-street map as our starting point, we might label it  $C_1$ , to emphasize that it is our “first cognitive structure.”

Experience will quickly require us to add more detail: we may soon encounter Erie Boulevard, another major east-west street, and we may find ourselves thinking of Syracuse as shown in Fig. 2, which we might label  $C_2$ .

Evidently, our second cognitive structure  $C_2$  is a refinement of  $C_1$ . In the language of Piaget, the process of “improving” the more primitive map  $C_1$  into the more sophisticated map  $C_2$  represents cognitive growth through “assimilation and accommodation” [see Appendix A, Flavell (114), p. 49, and elsewhere].

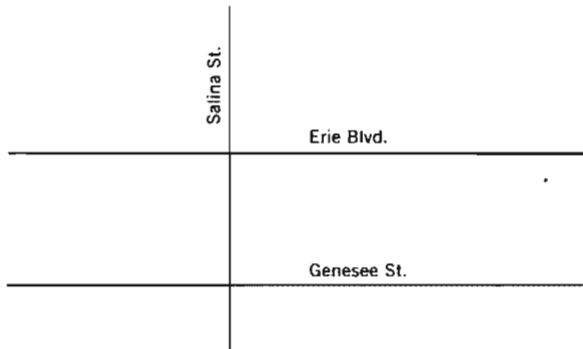


FIGURE 2

Sometimes we encounter situations where a modest change in the learner’s cognitive structure will not be sufficient. A more drastic change may be needed. We may discover, for example, that Erie Boulevard and Genesee Street, which we have been thinking of as roughly parallel, actually *intersect* on the east side of town. An agonizing reappraisal of our cognitive structure for the street layout of Syracuse is now in order. The map  $C_2$  must be discarded, to

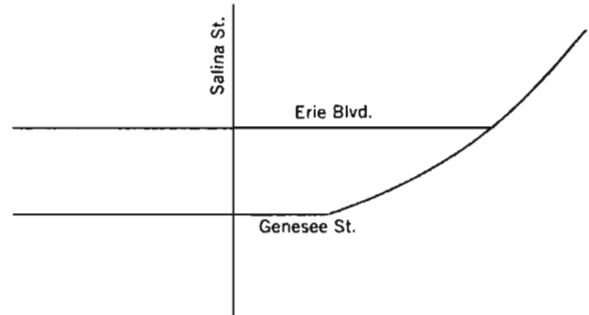


FIGURE 3

be replaced by  $C_3$ , which perhaps looks like the map in Fig. 3. This process of discarding  $C_2$  and replacing it by  $C_3$  represents, then, a more drastic instance of cognitive growth [see Appendix A, Flavell (114), Chapter 2].

If we use this general point of view, we may say that the child comes to each lesson with some cognitive structure  $C_n$ , and, as a result of the lesson, we expect that cognitive growth by assimilation and accommodation will occur, and the child will replace  $C_n$  by  $C_{n+1}$ . Teachers do not always think of learning in these terms, but this approach seems particularly appropriate as a way to think about the process of learning mathematics.

Notice that every cognitive structure is *wrong*; you cannot make from memory a perfect map of the city you live in, nor can you give, from memory, a perfect description of the total contents of your own house. “Right” and “wrong” are not useful criteria in judging cognitive structures—all cognitive structures are wrong. However, some are nonetheless preferable to others. In particular, a sequence of maps which grow like that in Fig. 4 is probably a less power-

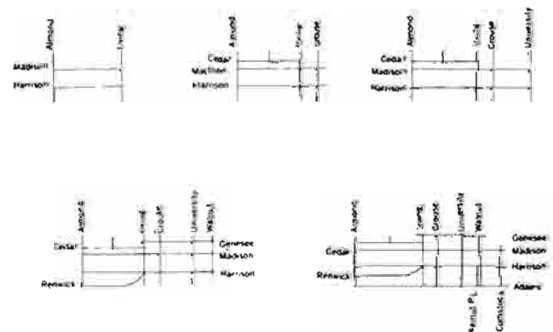


FIGURE 4



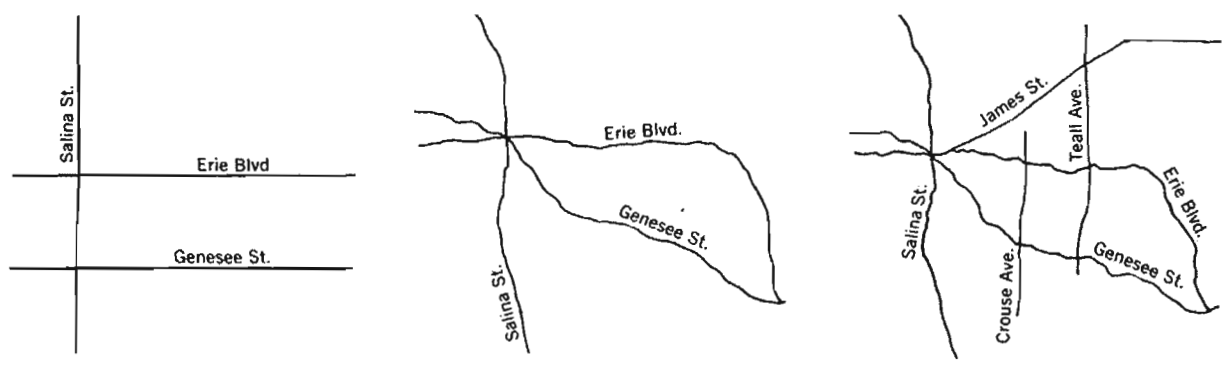


FIGURE 5

ful sequence than one which grows by the use of significant structural elements, somewhat like that in Fig. 5, even though both sequences are tending toward the same sophisticated map, shown in Fig. 6.

What does this say in the case of mathematics? If we devote grade 1 to addition facts up to 10, grade 2 to addition facts up to 100, and so on, we are putting one foot in front of the other, left, right, left, right, ... This approach is weak in power.

If, instead, we seek those basic mathematical concepts, techniques, and attitudes which play important structural roles in the development of the subject, we have a far more powerful approach. Cartesian coordinates, introduced (say) at grade 2, give us an ability to relate any arithmetic or algebraic problem to a geometric one, and vice versa. For all the rest of our lives we shall be able to unify algebra and geometry into a single coherent subject! This is power.

Once we learn such basic structural concepts as variable, function, mapping, and so on, we have a strong structural framework to which all of our subsequent mathematical learning can be related. This, again, is power! Here we are building cognitive structures that can well serve as foundations for improved structures in the future.

Piaget's notion of cognitive growth by assimilation and accommodation, his view of learning as a sequence of cognitive structures,

$$\dots, C_n, C_{n+1}, C_{n+2}, C_{n+3}, \dots$$

gives us a theoretical position from which we can answer, at least in part, the question of what should we teach. We should help the child build cognitive structures from which future cognitive structures can easily and powerfully emerge. It appears that a cognitive structure has more good growth potential if it is organized around concepts which play basic

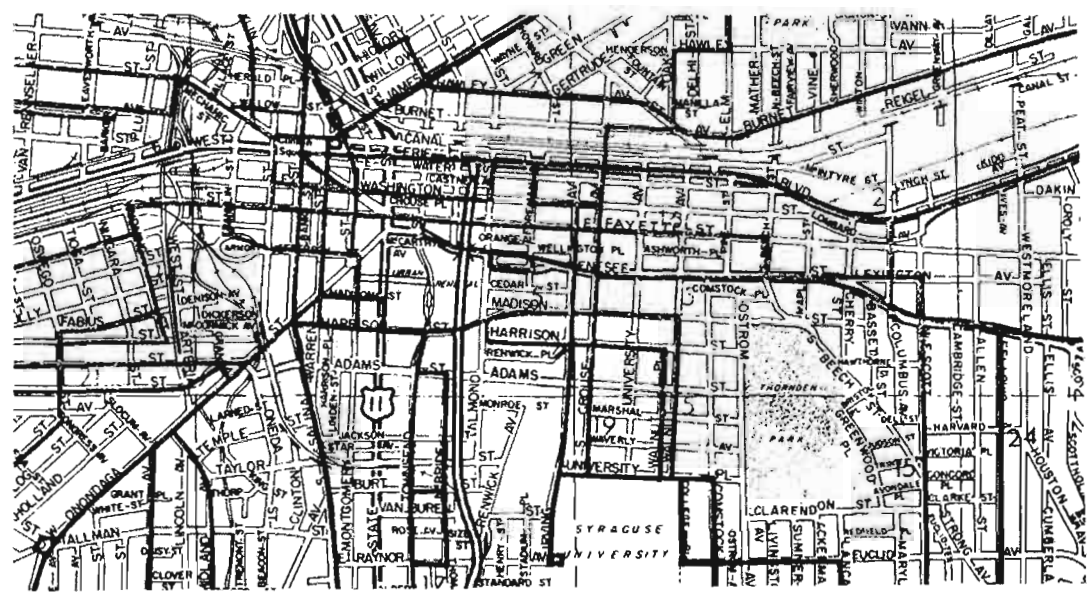


FIGURE 6

structural roles in mathematics as a whole — this provides a good framework for the assimilation of finer detail and provides maximum continuity and flexibility when the more drastic process of accommodation forces us to replace some cognitive structure with another quite different.

The mathematical concepts in this book — variable, open sentence, truth set, function, mapping, number line, Cartesian coordinates, implication, contradiction, and so on — have been chosen because we believe these concepts are of precisely this nature; they do play fundamental structural roles in the process of thinking about mathematics.

## *The topics in brief preview*

Which “new topics” are offered in the present volume? Primarily these:

**Logic.** There is a special branch of mathematics that is concerned with *combining statements*. This subject is known as *mathematical logic*. The importance of this topic has grown rapidly since 1900, to the point where its inclusion in precollege education seems nowadays to be extremely desirable. Surprisingly enough, mathematical logic is a relatively new subject, being (in its present form) a product mainly of the twentieth century. It promises to have important implications in any subject where *we start with some statements and work out others*, as we do in law, in philosophy, in science, and in mathematics.

**Empirical Statistics.** Some phenomena are properly thought of as “determined” or “deterministic.” For example, you hold a stone, release it, and it falls to the ground. It doesn’t sometimes fall — it always falls.

Other phenomena are not fully determined. For example, you have a headache, take two aspirin, and *perhaps* the sensation of pain ceases. Then again, perhaps it does not.

There is, however, a certain measure of regularity and consistency even within these “chance” phenomena. This regularity can often be studied, and even made use of, by means of statistics. This subject, also, has grown tremendously in importance in recent years. Its inclusion in precollege mathematics seems almost inevitable, probably at

many different grade levels, and in many different forms. In this book there is one chapter which introduces some of the ideas involved in the study of “random” variation, in relation to the question of measurement error.

**Basic Ideas about Numbers and Variables.** The study of statistics, logic, and similar topics is not possible unless one also learns about certain concepts that are basic to virtually all mathematics. The “essential” concepts include the arithmetic of signed numbers, the idea of variables, the idea of open sentence and truth set, the idea of functions, the idea of implication, and so on. (These matters are treated both in *Explorations* and also in *Discovery*.)

**Cartesian Coordinates.** The great invention of rectangular coordinates by René Descartes, in the seventeenth century, made it possible to unify algebra and geometry into a single subject. Being a seventeenth-century discovery, analytic geometry is clearly not “new,” but it is of fundamental importance, and many aspects of Descartes’ idea are actually quite simple. The subject is of great significance for the applications of mathematics in physical science, in social science, and elsewhere. Some of these applications have been taught easily and successfully at the kindergarten level; many are highly suitable for grades 4-8.

**Matrices.** Matrices are basic to a great deal of modern mathematics, and in many applications. In Project experimentation, matrices have proved to be a source of intrigue and a fruitful field for exploration by fifth-, sixth-, and seventh-grade children, as well as by students in senior high school.

**“Mappings” or “Transformations.”** This concept, which is presented in an elementary way in some of the following chapters, is one of the fundamental concepts of modern mathematics. If children learn it early in life, it can serve as part of the framework to which they can attach many other new ideas that will come along later.

**Derivation of the Quadratic Formula.** This topic, quite traditional at grade 9, has proved easy and interesting for children in grades 5, 6, 7, and 8. We include it here partly to relate our “new” mathematics to the “traditional” school program. (Our approach to the quadratic formula is, we hope, more creative than the traditional approach, and productive of deeper insights into what is really involved.)

*All of these topics will be explained at greater length in the appropriate following chapters.*

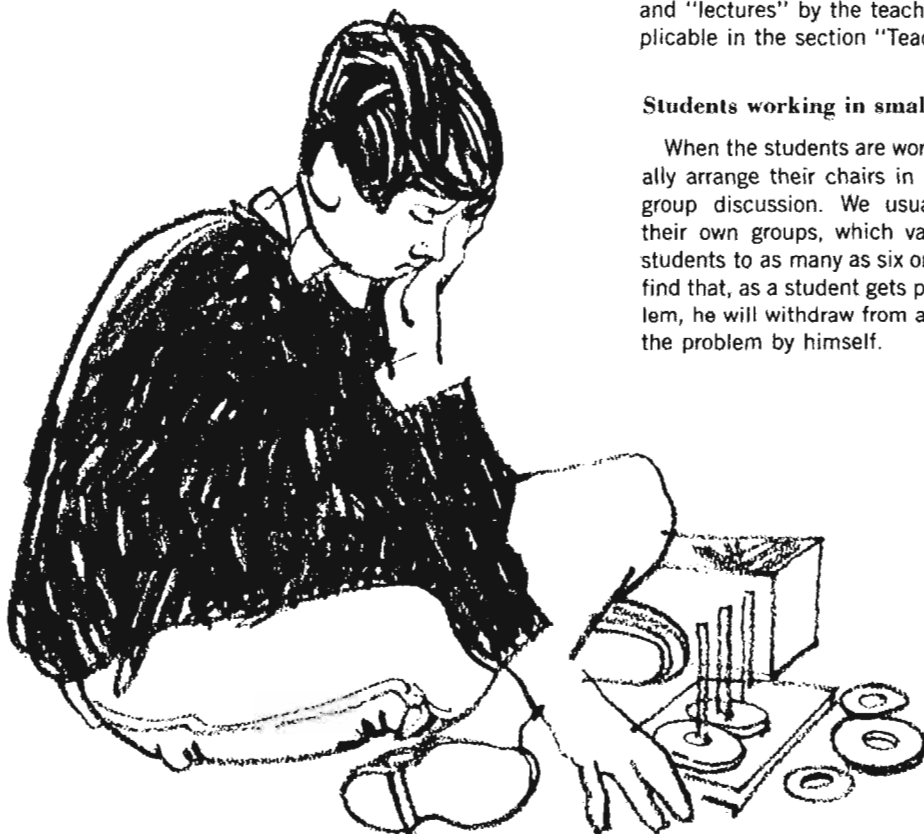


# PLANNING THE LESSONS

## *Using the Madison Project films*

Teaching the contents of this book is made easier by the existence of important forms of assistance for the teacher. Most important among these are the motion picture films showing actual classroom lessons. These films enable the teacher to see exactly how matrix algebra can be introduced to fifth-graders, how Cartesian coordinates can be introduced to second-graders, and so on. In many cases the same children can be followed, via the filmed lessons, for as long as five consecutive years of study of Madison Project materials. Consequently, teachers can see what their work will lead to in the next few years of the students' lives.

Notice that the original purpose of these films is to help teachers plan (and execute) successful lessons. The films are primarily intended for viewing by teachers, rather than by children. In a few cases, however, the films have been viewed by the children, with seemingly good results. Explore this possibility further if you wish. (A list of presently available films is given in Appendix B.)



## *A varied diet of experiences*

When mathematics is always a matter of reading, writing, and reciting, it cannot help but become dull for most students. Greater variety in the kinds of classroom experiences that they have with mathematics can change the attitudes of many children almost beyond belief.

For present purposes, we can distinguish four kinds of classroom experiences:

- (1) Students working in small groups, using the book.
- (2) Students working in small groups, performing some physical experiment.
- (3) Class discussion, led by the teacher (which may involve use of the book, or work on some physical experiment, or neither).
- (4) A "presentation" by the teacher, usually in the form of an informal "lecture," punctuated by student questions or other student remarks.

The following sections offer some suggestions for providing for the first two sessions and for combining and sequencing these four kinds of sessions. Class discussions and "lectures" by the teacher will be discussed when applicable in the section "Teaching the Materials."

### **Students working in small groups, using the book**

When the students are working in small groups, they usually arrange their chairs in clusters for convenient small-group discussion. We usually let the students choose their own groups, which vary in size from two or three students to as many as six or eight students. We frequently find that, as a student gets particularly interested in a problem, he will withdraw from a group discussion and work on the problem by himself.

When these sessions are going really smoothly, the teacher appears to have almost nothing to do. Actually, by this we mean that he may not be called upon for a single instance of exposition or explanation. His presence presumably helps keep schoolwork in sight, and prevents degeneration into a purely social party situation. Moreover, the fact that the teacher himself obviously *values* mathematics enough to have spent a large part of his life thinking about it tends to influence children profoundly. But the teacher's role is one of transmitting values and setting "tone." He does not actually need to "explain" anything.

(In order to emphasize this matter of values, Project teachers will frequently sit at their desks and work out some of the harder problems themselves. There seems to be no more effective way to show that you value a task than to actually do it!)

When things are going less smoothly, the teacher has a more active role to play, in terms of more explicit interactions with students. These may take many forms: talking with individual students or working with one group or working briefly with the entire class. The teacher may suggest alternative problems or topics for students to work on; explain the general purpose of the lesson if that is not clear; solve a problem, by way of illustration; exhibit the work of one student, for discussion by the entire class; summarize what has been done; recall earlier methods that may be needed and so forth.

In lessons of this sort, we generally think of the teacher as playing the role of a foreman with a group of workers.\*

Although this role for the teacher has been known to many fine teachers for some time, it is relatively new for most Madison Project teachers, and we are quite excited about it. When the teacher is cast in this "foreman" role, how can he be most effective? When should he take an interest in the work of a student? When should he avoid interfering? Should he avoid looking out of the window, on the grounds that this overemphasizes his special prerogatives, and perhaps disparages the importance of the task the students are working on? When should he sit at his desk, and when should he walk around the room? When he joins one of the small groups, is it important for him to sit at a student chair or desk, in order to seem a member of the group, working on a common problem?

All of the usual maxims for good teaching presumably apply to the "foreman" role also: respect the student as a human being and as an equal; be careful to avoid condescending to the student; listen—really listen!—to what the student says; when exposition is called for, express the *essentials*, not the minor details; and, perhaps above all, *avoid talking too much, doing too much, or interfering too much.*

\*Nothing invidious is meant by this. Remember that there are good foremen and poor ones, good work situations and poor ones. A "foreman" is not necessarily unpleasant or authoritarian; "work" can build growth, and even be quite enjoyable. Not all "work situations" are deadly and exploitative.

With ninth-grade classes, we usually arrange to have both *Discovery* and *Explorations* available at these sessions. Each student has his own personal book and writes in it. We believe students show a marked improvement in interest when each is allowed to write in his own book.

**Students working in small groups, performing a physical experiment**

In general, this situation is similar to that just discussed. We would, however, like to emphasize the importance of letting students perform physical experiments in the mathematics class. This involves some extra effort, in arranging materials and equipment, but *the improvement in motivation is almost beyond belief.*

**Combining and sequencing classroom experiences**

In general, working in small groups is especially valuable in providing experience ("building readiness") for subsequent class discussions or informal lectures. For example, we let ninth-graders work in small groups on Chapters 32, 33, and 34, with some brief class discussion. After several days of this, we work out the derivation of the quadratic formula on the chalkboard, with total class discussion. The gains in this discussion, as a result of the few days of previous "individualized exploration," are dramatically apparent.

Our sequence—explore, then discuss—might be represented diagrammatically, as in Fig. 7.

Sometimes small-group work can stand entirely alone. If it can, we would suggest that the teacher refrain from adding on any unnecessary "summary," which somehow appears condescending—as if the students can't think for themselves. For ninth-graders, many topics usually go perfectly smoothly, with no help (or interference) from the teacher. (We allow students considerable freedom in "jumping around" in the book as they wish, and do not require them to work every problem. Nor need all students work on the same chapter at the same time, except where total class discussion makes this necessary.)

Working in small groups gives opportunities for review or restudy by those individual students who seem to need

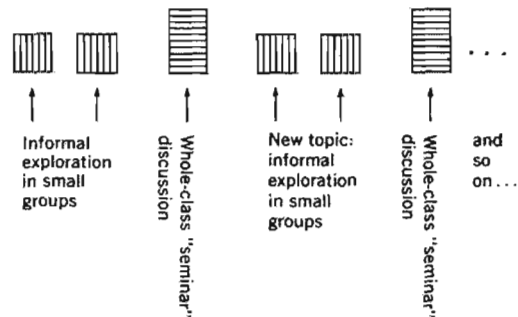


FIGURE 7

it. Around midyear, some individual students may take advantage of small-group sessions to work through the chapters that deal with "postman stories" and the arithmetic of signed numbers.

In the following section we shall comment briefly on which of the four types of lessons seem to work best for each individual topic. There is, of course, considerable variation from one class to another.

### Some remarks on grade level

Hopefully, we are moving toward the day when the artificial lines of demarcation known as "grade levels" will disappear, and each child will really pursue a course of learning that is meant just for him, *and which he helps to determine*.

That day is not here, for most of us, as yet. Consequently — with apologies — we indicate here some of the experiences the Project has had in teaching the various chapters and topics at different grade levels.

At the same time, we indicate which of the four types of classroom lessons we usually use for each topic.

Chapter	Suitable Grade Levels	Type of Lesson
Chapter 1		
<input type="checkbox"/> notation	grade 5 - grade 9	Class discussion or small groups working from the book.
Replacement set	grade 5 - grade 9	Class discussion or small groups working from the book.
Use of statements as replacements for variables	grade 5 - grade 9	Class discussion or small groups working from the book.
Chapter 2		
Cartesian product of two sets	grade 5 - grade 9	Class discussion or small groups working from the book.
Plotting points on Cartesian coordinates	grade 5 - grade 9	Class discussion or small groups working from the book.

Tic-Tac-Toe Game	grade 5 - grade 9	Class divided into two teams; you may want to present this without using the book.
Chapter 3		
Open sentences with two variables	grade 5 - grade 9	Class discussion (with students volunteering to come to chalkboard) or small groups working individually with the book.
Chapter 4		
Introduction of signed numbers, using the "pebbles-in-bag" model	grade 5 - grade 9	Probably most effectively done as class discussion, as in the film "A Lesson with Second Graders."
Chapter 5		
Arithmetic of signed numbers, using "postman stories" (sums and differences)	grade 5 - grade 9	Probably most effectively done as class discussion.
Chapter 6		
Arithmetic of signed numbers, using "postman stories" (products)	grade 5 - grade 9	Class discussion or small groups working from the book.
Chapter 7		
Nonstandard numerals and nonstandard algorithms	The appropriate grade levels for this topic will vary from school to school, depending upon the basic arithmetic program that is used	Class discussion or small groups working from the book.
Chapter 8		
Graphs with signed numbers	grade 5 - grade 9	Class discussion or small groups working from the book.

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Chapter 9 PN and UV	grade 5 - grade 9	Might be recommended for reading outside of class; alternatively, use either class discussion or small groups.	Is It An Identity?  Recognizing identities	grade 5 - grade 9	We use this as a game between two teams, with the entire class participating, but you may find better methods. Notice that this <i>does not</i> appear in the student book at all! We let the teacher introduce the game.
Chapter 10 The "secrets" (i.e., the coefficient rules) for quadratic equations from the book	grade 5 - grade 9	Class discussion or small groups working with books (cf. the films "First Lesson" and "Second Lesson").			
Chapters 11 - 15 Logic	We prefer this for grade 7 or later; you might be able to use it satisfactorily with somewhat younger children.	Class discussion or small groups working from the book.	Chapter 19 Describing the process of obtaining new identities from old ones, using PN and UV	grade 5 - grade 9	Class discussion or small groups working from the book.
Chapter 16 The Game of Clues	grade 5 - grade 9 (If you keep it simple, this game is fun even for first graders. Don't hesitate to change the rules, in order to make it really simple.)	The best method is to ignore the book and present this as a game to be played in class, with the teacher explaining the rules.	Chapter 20 Shortening lists of algebraic statements by using <i>generalization</i> and <i>implication</i>	grade 5 - grade 9	Class discussion or small groups working from the book.
Chapter 17 Measurement uncertainties (an introduction to some ideas of statistics)	We like this lesson with sixth-graders, but grades 7, 8 and 9 would be suitable. Indeed, it is an excellent lesson for ninth-graders.	This is a "physical experiment" type of lesson (cf. the film "Average and Variance").	Chapter 21 Writing derivations	grade 5 - grade 9 (We usually prefer either grade 5, or grade 6, or grade 9.)	Possibly it is best to let the students work from the book, in small groups, and <i>then</i> have a "summing up" session where the derivations are made in a total class discussion.
Chapter 18 Identities	This would seem to be an essential part of the mathematics program by the time you reach grade 9.	Class discussion or work in small groups or as individuals.	Chapter 22 Extending our axiom system, to provide for subtraction and division.	grade 5 - grade 9	Class discussion or small groups working from the book

Chapter 23 Making up derivations	grade 5 - grade 9	Class discussion or small groups working from the book.	Chapter 29 Transform operations and "equivalent equations"	grade 5 - grade 9 (This might also be appropriate for younger children.)	Our favorite method is general class discussion, but the participation in the class discussion might be better if, <i>before</i> the class discussion, the students worked in small groups, using their books.
Chapter 24 Extending systems: "lattices" and exponents	grade 5 - grade 9	Class discussion; alternatively, let students work in small groups, using their books. (You may want a class discussion near the end.)	Chapter 30 Transform operations on inequalities	grade 5 - grade 9 (This might also be appropriate for younger children.)	For this one case, it may be well to have the class discussion <i>first</i> , and then let the students work in small groups, using their books.
Chapter 25 Guessing functions	grade 5 - grade 9	This is probably best done as a class activity, as in the film entitled "Guessing Functions." It may well be that you will prefer not to use the student books at all on this chapter.	Chapter 31 "Variables" versus "constants"	grade 5 - grade 9	Probably introduction by the teacher is most effective, presumably to the entire class; alternatively, let students work in small groups, using their books.
Chapter 26 The form of functions (linear, quadratic, etc.)	grade 9	Class discussion or students working from their books (perhaps in small groups).	Chapter 32 General solution of equations	grade 5 - grade 9	Students working in small groups, using their books or working individually, using their books.
Chapter 27 Physical experiments that lead to functions	grade 5 - grade 9	Physical experiments performed in class, preferably with the students working in small groups.	Chapter 33 Hints on problem solving, à la Polya	grade 9 (Possibly suitable also for some younger children.)	Class discussion or small groups working from the book.
Chapter 28 The notation $f(x)$	grade 6 - grade 9	Small groups working from the book.			

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Chapter 34 The quadratic formula	This is a traditional topic in grade 9, and in our experience that is a very good grade level for it; however, we have used it successfully with bright students in grades 6, 7, and 8.	At the ninth-grade level, we prefer first to have the students work in small groups, using the book. <i>After</i> this, we go through the entire derivation in a discussion with the entire class.	Chapter 39 Exploring matrices, continued	grade 5 - grade 9 (Perhaps Chapter 39 is <i>easier</i> than Chapter 38; you may prefer to omit Chapter 38 and go directly from Chapter 37 to 39.	Either small groups or the entire class.
Chapter 35 History	grade 9 (Possible also in grades 5 and 6)	You can assign this for reading outside of class.	Chapter 40 Fun with matrices and transformations	grade 5 - grade 9	Perhaps small groups or students working as individuals. When students make up an interesting mapping, they can solve it with the entire class.
Solving cubic equations by Cardano's method	grade 9	Students working in small groups, followed by general class discussion.	Chapter 41 Matrices and transformations	grade 5 - grade 9	Either small groups or class discussion.
Chapter 36 Mappings	grade 5 - grade 9	Combine some small-group work with class discussion. These are fun and easy, <i>once you see the point</i> .	Chapter 42 Matrix solution of simultaneous equations	grade 9	Small groups.
Chapter 37 Introduction to matrix multiplication	grade 5 - grade 9	We prefer to have the teacher introduce this, working with the entire class.	Chapter 43 Word problems	grade 9	Small groups.
Chapter 38 Exploring matrices	grade 5 - grade 9	Class discussion or small groups working from the book. You may want to use some of each approach.	Chapter 44 Establishing an isomorphism between a special set of matrices and the set of rational numbers	grade 5 - grade 9	Small groups or class discussion.
			Chapter 45 $x^2 = -4$	grade 5 - grade 9	Small groups or class discussion.
			Chapter 46 Determinants	grade 9	Small groups or class discussion.
			Chapter 47 Finding matrix inverses	grade 9	Small groups or class discussion.



## TEACHING THE MATERIALS

This section contains a sequence of "informal exploratory experiences" that, taken together, provide an introduction to such mathematical topics as matrices, logic, functions, and Cartesian coordinates. In the following pages you will find the *Student Discussion Guide* reprinted chapter by chapter for your convenience. It appears in the left-hand column of the page. Opposite each exercise of the *Student Discussion Guide*, in the right-hand column, is the answer to the exercise along with helpful comments and suggestions for teaching the material, when deemed necessary. Very often, also, you will find chapter background or introductory information preceding the Answers and Comments.

You have, therefore, in one convenient location, the *Student Discussion Guide*, the answers for every exercise, helpful teaching suggestions, and mathematical background material.

## PART ONE • VARIABLES, GRAPHS, AND SIGNED NUMBERS

chapter 1 / Pages 1-4 of Student Discussion Guide

### VARIABLES

One can make very little progress in mathematics without the concept of *variable*. When a mathematician says "variable," he is referring to the concept which is usually written as an

$x, y, N, a, t, r, \square, \triangle, \nabla$ , etc.

We present here, *for teachers*, a somewhat more formal discussion of this concept than we would usually present to children.

The list of statements\*

$$(3 \times 3) + 1 < 20$$

$$(4 \times 4) + 1 < 20$$

$$(5 \times 5) + 1 < 20$$

$$(6 \times 6) + 1 < 20$$

$$(10 \times 10) + 1 < 20$$

contains two true statements (the first two) and three false statements (the last three). All five of these statements may be represented by one single mathematical "sentence" involving a variable, as follows:

$$(\square \times \square) + 1 < 20 \quad (1)$$

By itself, however, equation (1) does not permit us to reconstruct the original five statements, for one simple reason: we do not know, merely from (1), which numbers (or, if you prefer, numerals) are to be written in the  $\square$ . In order to reconstruct the first statement, we must write 3 in the  $\square$ .

Mathematicians describe this by saying, "Use three as a replacement for the variable  $\square$ ." We shall call this process "UV" (which stands for "use of a variable"), and shall write

$$3 \rightarrow \square,$$

If we carry out the operation

$$3 \rightarrow \square,$$

we get

$$(\boxed{3} \times \boxed{3}) + 1 < 20,$$

or, if you prefer,

$$(3 \times 3) + 1 < 20.$$

\*Notice that the symbol  $<$  means "is less than." Consequently, the statement " $3 < 5$ " is true, whereas the statement " $10 < 7$ " is false.



In order to reconstruct *all five* of the original statements from equation (1), we must perform

$$3 \rightarrow \square,$$

and then, going back to (1), perform

$$4 \rightarrow \square,$$

and so on, for 5, 6, and 10. Mathematicians indicate all five of these uses of UV by saying that *the replacement set\** for the variable  $\square$  is

$$\{3, 4, 5, 6, 10\}.$$

We might name this set "R," for "replacement," and write

$$R = \{3, 4, 5, 6, 10\}. \tag{2}$$

Now equations (1) and (2) together permit us to reconstruct our original list of five statements. That is to say, the list

$$\begin{aligned} (\square \times \square) + 1 < 20 \\ R = \{3, 4, 5, 6, 10\} \end{aligned}$$

says exactly the same thing as this list:

$$\begin{aligned} (3 \times 3) + 1 < 20 \\ (4 \times 4) + 1 < 20 \\ (5 \times 5) + 1 < 20 \\ (6 \times 6) + 1 < 20 \\ (10 \times 10) + 1 < 20 \end{aligned}$$

Sometimes, however, we want to go beyond merely *listing* these five statements; sometimes we want to label them appropriately as true or false:

$(3 \times 3) + 1 < 20$	True
$(4 \times 4) + 1 < 20$	True
$(5 \times 5) + 1 < 20$	False
$(6 \times 6) + 1 < 20$	False
$(10 \times 10) + 1 < 20$	False

\*In mathematics, the word *set* means merely a collection or an aggregate. We often write sets by using *braces* or "wiggly brackets," so that, for example, the set of New England states might be written

$$\{\text{Maine, New Hampshire, Vermont, Massachusetts, Connecticut, Rhode Island}\}.$$

Or the set of even numbers less than ten and greater than zero might be written

$$\{2, 4, 6, 8\}.$$

The order of writing the elements of a set is considered to be irrelevant, so that this last set could also be written

$$\{6, 2, 4, 8\}.$$

To do this, using our powerful new shorthand, we shall use  $T$  to name the set of those elements of  $R$  which produce a *true* statement, and  $F$  to name the set of those elements of  $R$  which produce a *false* statement. Evidently, in the present example,  $T = \{3, 4\}$  and  $F = \{5, 6, 10\}$ . Using this notation, the list

$$(\square \times \square) + 1 < 20$$

$$R = \{3, 4, 5, 6, 10\}$$

$$T = \{3, 4\}$$

$$F = \{5, 6, 10\}$$

says exactly the same thing as this list:

$$(3 \times 3) + 1 < 20 \quad \text{True}$$

$$(4 \times 4) + 1 < 20 \quad \text{True}$$

$$(5 \times 5) + 1 < 20 \quad \text{False}$$

$$(6 \times 6) + 1 < 20 \quad \text{False}$$

$$(10 \times 10) + 1 < 20 \quad \text{False}$$

The power of this new shorthand is very great. Indeed, the "golden period" of mathematical advance was unlocked, in part, by the introduction of the concept of variable. (Notice that, unfortunately, the word *variable* is treacherous, for its mathematical meaning is quite different from its everyday meaning. Max Beberman of UICSM has preferred the name "pro-numeral," by analogy with "pronoun.")

A mathematical sentence which involves a variable is called an *open sentence*, consequently,

$$3 + \square = 5$$

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$$(\square \times \square) + 1 < 20$$

are all examples of open sentences. All mathematical sentences are divided into *open sentences*, *true statements*, and *false statements*.

The set  $T$  is called the truth set for the open sentence: that is, it is the set of those elements of  $R$  which will produce a true statement when we use UV.

In using UV—that is to say, in making numerical replacements for a variable, as we did when we used

$$3 \rightarrow \square$$

to go from

$$(\square \times \square) + 1 < 20$$

to

$$(3 \times 3) + 1 < 20,$$

and so on—it is important to obey an agreement which mathemati-

cians all make: whatever number is written in the first  $\square$  in the sentence, *that same number must be written in every other  $\square$  in the sentence.* This agreement makes the concept of variable useful; without it, variables could not play their present important role in mathematics. We shall refer to this agreement as the "rule for substituting."<sup>\*</sup>

Frequently we shall not bother to indicate the replacement set  $R$ . In such cases, the reader is free to use *any number he knows about* as a replacement for the variable.

We would never present a formal discussion such as the preceding one, to children. Rather, when we want children to meet the concept of variable, we shall try to arrange for them to meet this concept *unobtrusively*, and to come upon it while they are in the process of doing something.

All right. Then we, as teachers, must ask ourselves what "active" thing children can do with variables. One answer, evidently, is that we can ask them to solve equations such as

$$3 + \square = 5$$

or

$$\square + \square + \square = 9,$$

and so on. This, then, is what we shall now do.



## CHAPTER 1 Variables

[page 1]

One of the major discoveries in mathematics occurred when someone<sup>\*</sup> realized that you could use sentences such as

$$3 + \square = 5.$$

A sentence like this is called an **open sentence** or an **equation**.

(1) Can you write something in the  $\square$  in the equation above so as to produce a **true statement**?

(2) Can you write something in the  $\square$  in the equation above so as to produce a **false statement**?

<sup>\*</sup>The original discovery of open sentences (and what we shall call variables) occurred so long ago that its precise origin is not known. An English scholar named A. Henry Rhind found, and bought, what turned out to be a very ancient Egyptian papyrus dating from before 1650 B.C. This papyrus is now the property of the British Museum. It was the work of a very ancient scribe named Ahmes, and it includes the basic idea of variables. However, surprising as it may seem, Ahmes and the people of his time did not have a really satisfactory method for writing numbers, and this appears to have prevented them from doing very much with the idea of variables. We, today, can do a great deal with this idea, and will undoubtedly continue to discover more and more ways to use variables.

## ANSWERS AND COMMENTS

(1) If you write 2 in the  $\square$ , you will get the statement

$$3 + \boxed{2} = 5$$

(or, if you prefer,  $3 + 2 = 5$ ) which is, of course, true.

(2) Any number other than 2 will produce a false statement. For example, 1965 will produce

$$3 + 1965 = 5 \quad \text{False.}$$

<sup>\*</sup>Compare *Discovery* (teachers' text), pages 25 and 26. You may also be interested in the film "First Lesson."

(3) Do you know what mathematicians mean when they speak of "the truth set for the open sentence  $3 + \square = 5$ "? Do you know how to write this truth set?

(3) The truth set for the open sentence  $3 + \square = 5$  is the set of numbers which will yield a true statement. In this case,

$$T = \{2\}.$$

(Notice that we have also just answered the question of how to write this.)

For most students this is, of course, a rhetorical question; they presumably do not know (however, having older brothers and sisters, they may!). In any case, we prefer asking this question, rather than merely making a statement; we feel the question does a good job of getting the students' attention.

(4) Do you know what mathematicians mean when they speak of the "rule for substituting"?

(4) The rule for substituting might be stated as "Whatever number you write in the first box in an open sentence, you must write this same number in all the other boxes."

(5) Bill claims that the "rule for substituting" says this: if an open sentence has more than one  $\square$  in it, you must put the same number in every occurrence of the  $\square$ . Do you agree?

(5) Bill is, of course, correct.

(6) Can you give some examples of using the "rule for substituting"?

(6) Here are a few:

If we start with the open sentence

$$\square + 0 = \square$$

and use UV,

$$3 \rightarrow \square,$$

we get

$$3 + 0 = 3.$$

If we use

$$1985 \rightarrow \square$$

we get

$$1985 + 0 = 1985.$$

(7) Tony says that the "rule for substituting" is not the same as "making a true statement." Do you agree?

(7) Tony is correct, as questions 8 through 12 clearly demonstrate.

(8) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to obey the rule for substituting but make a false statement?

(8) There are many ways to do this; here is one:

$$\boxed{7} + \boxed{7} + \boxed{7} = 9, \quad \text{or} \quad 7 + 7 + 7 = 9.$$

We have put the same number in each  $\square$ , and surely our resulting statement is false, so we have successfully done what the question asked us to do. (You may find it helpful to view the film "First Lesson.")

- (9) For the open sentence [page 2]

$$\square + \square + \square = 9,$$

can you substitute so as to **violate** the rule for substituting but make a **true** statement?

- (10) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **violate** the rule for substituting and make a **false** statement?

- (11) For the open sentence

$$\square + \square + \square = 9,$$

can you substitute so as to **obey** the rule for substituting and make a **true** statement?

- (12) What is the
- truth set**
- for the open sentence

$$\square + \square + \square = 9?$$

Can you find the **truth set** for each open sentence?

(13)  $\square + \square = 12$

(14)  $(2 \times \square) + 1 = 7$

- (9) Again there are many possibilities; here are a few:

$$\boxed{4} + \boxed{4} + \boxed{1} = 9$$

$$\boxed{2} + \boxed{3} + \boxed{4} = 9$$

$$\boxed{3} + \boxed{1} + \boxed{5} = 9$$

$$\boxed{8} + \boxed{1} + \boxed{0} = 9$$

$$\boxed{2} + \boxed{2} + \boxed{5} = 9$$

and so on.

- (10) Here, also, there are many possibilities; here are some of them:

$$\boxed{1} + \boxed{2} + \boxed{3} = 9$$

$$\boxed{1066} + \boxed{1732} + \boxed{1985} = 9$$

$$\boxed{1} + \boxed{1} + \boxed{0} = 9$$

$$\boxed{\frac{1}{2}} + \boxed{\frac{1}{3}} + \boxed{\frac{1}{2}} = 9$$

- (11) There is only one way to do this:

$$3 \rightarrow \square$$

$$\boxed{3} + \boxed{3} + \boxed{3} = 9$$

- (12)
- $\{3\}$

- (13)
- $\{6\}$

- (14)
- $\{3\}$

Younger students, especially, will usually attack this problem by trial and error, which is a good method in this example. For instance, if we try  $2 \rightarrow \square$ , we find

$$(2 \times 2) + 1 = 7$$

$$4 + 1 = 7$$

$$5 = 7,$$

which is evidently *false*. Moreover, we see that 2 is *too small*.

Hence, let's try  $3 \rightarrow \square$ :

$$(2 \times 3) + 1 = 7$$

$$6 + 1 = 7$$

$$7 = 7,$$

which is *true*.

Since  $3 \rightarrow \square$  yielded a *true statement*, the *truth set* must be  $\{3\}$ .

Actually, one might ask if there aren't some other numbers that will also work. This is, *a priori*, a possibility; however, in problems of the present type, the truth set contains exactly one element, so we have completed our work on this problem once we have found that  $3 \rightarrow \square$  produces a true statement.

$$(15) \quad (3 \times \square) + 5 = 38$$

$$(15) \quad \{11\}$$

$$(16) \quad (2 \times \square) + 20 = 25$$

$$(16) \quad \{2\frac{1}{2}\}$$

Here, the trial-and-error method which we discussed in answer to question 14 can be used to show us that 2 is too small and also that 3 is too large. For a further discussion of problems like this, see *Discovery*, Chapters 20, 21, and 22. You may also wish to view the film entitled "Open Sentences and the Number Line."

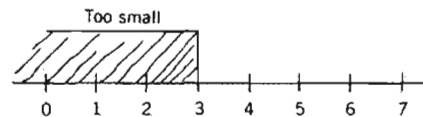
$$(17) \quad (3 \times \square) + 11 = 22$$

$$(17) \quad \{3\frac{2}{3}\}$$

Children usually tackle this problem something like this:

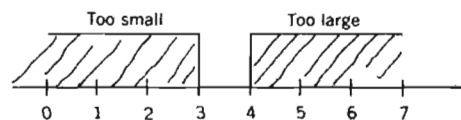
$$\begin{aligned} \text{Try } 3 &\rightarrow \square: \\ (3 \times 3) + 11 &= 22 \\ 9 + 11 &= 22 \\ 20 &= 22 \quad \text{False} \\ 3 &\text{ is too small.} \end{aligned}$$

We can represent this on a *number line*:



$$\begin{aligned} \text{Try } 4 &\rightarrow \square: \\ (3 \times 4) + 11 &= 22 \\ 12 + 11 &= 22 \quad \text{False} \\ 4 &\text{ is too large.} \end{aligned}$$

The number-line picture now looks like this:



Since  $3 \times \square$  must be a whole number, we can see what denominator we must use. Evidently, the answer must be either  $3\frac{1}{3}$  or  $3\frac{2}{3}$ , or else no number works! Which is it?

Try  $3\frac{2}{3} \rightarrow \square$  :

$$(3 \times 3\frac{2}{3}) + 11 = 22$$

$$11 + 11 = 22 \quad \text{True}$$

Consequently,

$$T = \{3\frac{2}{3}\}.$$

(18)  $(10 \times \square) + 1 = 28$

(18)  $\{2\frac{2}{10}\}$  or  $\{2.7\}$

The procedure here is generally similar to that of the preceding problem.

(19) What do we mean (in mathematics) when we speak of a variable? How can we write a variable? How many ways do you know to use a variable?

(19) This, too, *may* be a rhetorical question. Still, one or more of the students may actually know that we write "variables" as  $x, y, a, A, r, \square, \triangle, \nabla, b, h$ , etc. These symbols behave much like pronouns. The sentence

He is President of the United States

or, even better,

\_\_\_\_\_ is President of the United States

has, itself, no truth value. It will become true or false depending upon what we put in the blank, or use as an antecedent of "he."

In just the same way,

$$\square + 9 = 15$$

is neither true nor false; it will become true or false when we use some number as a replacement for the  $\square$ . The  $\square$  is known as a *variable*.

Alternatively, we can think of

$$\square + 9 = 15$$

with the replacement set

$$R = \{1, 2, 3, 4, 5, \dots\}$$

as standing for the unending list of statements

$$1 + 9 = 15$$

$$2 + 9 = 15$$

$$3 + 9 = 15$$

$$4 + 9 = 15$$

⋮

One of these statements is true; all of the others are false.

(20) Sometimes, when we want to be very precise, we specify the replacement set for the variable. John says that if we specify the replacement set  $R = \{3, 4, 5, 6, 7\}$  for the variable  $\square$  and if we write the open sentence

$$3 + \square = (2 + \square) + 1,$$

we have really written five mathematical statements that don't have  $\square$ 's in them! Do you agree?

(20) John is right. The open sentence and replacement set

$$3 + \square = (2 + \square) + 1$$

$$R = \{3, 4, 5, 6, 7\}$$

mean *exactly the same thing* as the five statements:

$$3 + 3 = (2 + 3) + 1 \quad 3 + 6 = (2 + 6) + 1$$

$$3 + 4 = (2 + 4) + 1 \quad 3 + 7 = (2 + 7) + 1$$

$$3 + 5 = (2 + 5) + 1$$

(21) Do you know what the symbol  
 $<$   
 means (in mathematics)?

(22) Which statements are true and which are false?

- (a)  $3 < 5$
- (b)  $(2 \times 3) + 1 < 12$
- (c)  $2 < 1$
- (d)  $0 < 7$
- (e)  $1000 < 1,000,000$
- (f)  $12 < 7$
- (g)  $1006 < 1060$
- (h)  $2000 < 1000$
- (i)  $5 < 5$
- (j)  $(3 \times 4) + 1 < 10$
- (k)  $(3 \times 4) + 1 < 13$

[page 3]

(23) Suppose you wanted to tell somebody these seven statements—only these seven, and no others:

- $5 + 4 < 21$
- $5 + 5 < 21$
- $5 + 6 < 21$
- $5 + 7 < 21$
- $5 + 8 < 21$
- $5 + 9 < 21$
- $5 + 10 < 21$

Could you do this by writing only one open sentence and by writing the replacement set for the variable  $\square$ ?

Since mathematics is deeply concerned with the discovery of pattern, notice how clearly the *pattern* of these statements is revealed by the variable notation.

(21) The symbol  $<$  means "is less than." For example, the statement

$$3 < 5$$

would be read:

Three is less than five.

(22) (a) True

(b) True

(c) False

(d) True

(e) True

(f) False

(g) True

(h) False

(i) False

(j) False

(k) False

It is often helpful to interpret

$$a < b$$

to mean

$a$  lies to the left of  $b$  on the number line.

We shall see the value of this interpretation when we begin to deal with signed numbers.

(23) The open sentence and replacement set

$$5 + \square < 21$$

$$R = \{4, 5, 6, 7, 8, 9, 10\}$$

mean exactly the same thing as this list:

- $5 + 4 < 21$
- $5 + 5 < 21$
- $5 + 6 < 21$
- $5 + 7 < 21$
- $5 + 8 < 21$
- $5 + 9 < 21$
- $5 + 10 < 21$



(24) Paul has written some open sentences and indicated replacement sets for each variable. In each case, can you write the statements Paul means, without using variables?

$$(a) (3 \times \square) + 1 < 25 \quad R = \{0, 2, 4, 6\}$$

$$(b) \square + \square = 2 \times \square \quad R = \{100, 7, 3, 24\}$$

$$(c) 3 + \square = \square + 3 \quad R = \{5, 7, 9, 10, 11\}$$

(25) Jill wrote these statements:

$$7 + (2 \times 1) < 50$$

$$7 + (2 \times 3) < 50$$

$$7 + (2 \times 4) < 50$$

$$7 + (2 \times 10) < 50$$

Can you indicate these four statements by writing one open sentence and by writing the replacement set for the variable?

(26) Don wrote these statements:

$$(8 + 1) \times (8 - 1) = (8 \times 8) - (1 \times 1)$$

$$(8 + 2) \times (8 - 2) = (8 \times 8) - (2 \times 2)$$

$$(8 + 3) \times (8 - 3) = (8 \times 8) - (3 \times 3)$$

$$(8 + 4) \times (8 - 4) = (8 \times 8) - (4 \times 4)$$

Can you represent Don's four statements by writing one open sentence and by indicating the replacement set for the variable?

[page 4]

(27) Nancy used the letter  $P$  to indicate a variable, and she wrote this for the replacement set for  $P$ :

$$R = \{\text{Jill, Eva, Eileen}\}$$

Then she wrote:

I like  $P$ .

What did Nancy mean?

$$(24) (a) (3 \times 0) + 1 < 25$$

$$(3 \times 2) + 1 < 25$$

$$(3 \times 4) + 1 < 25$$

$$(3 \times 6) + 1 < 25$$

$$(b) 100 + 100 = 2 \times 100$$

$$7 + 7 = 2 \times 7$$

$$3 + 3 = 2 \times 3$$

$$24 + 24 = 2 \times 24$$

$$(c) 3 + 5 = 5 + 3$$

$$3 + 7 = 7 + 3$$

$$3 + 9 = 9 + 3$$

$$3 + 10 = 10 + 3$$

$$3 + 11 = 11 + 3$$

You may want to view the film entitled "Variables and Replacement Sets."

$$(25) 7 + (2 \times \square) < 50$$

$$R = \{1, 3, 4, 10\}$$

$$(26) (8 + \square) \times (8 - \square) = (8 \times 8) - (\square \times \square)$$

$$R = \{1, 2, 3, 4\}$$

(27) The open sentence and the replacement set

I like  $P$ .

$$R = \{\text{Jill, Eva, Eileen}\}$$

mean exactly the same thing as this list:

I like Jill.

I like Eva.

I like Eileen.

Note: Sometimes, in order to be sure we match up the replacement set with the variable correctly, we shall use subscripts, like this:

$$R_p = \{\text{Jill, Eva, Eileen}\}.$$

(28) Tom used the letter  $P$  to indicate a variable, and he said that the replacement set for the variable  $P$  was to be:

$$R = \{ \text{"St. Louis is a city", "New York is a city", "Los Angeles is a city", "Miami Beach is a city", "Minot is a city"} \}$$

Then Tom wrote:

$P$  is a true statement.

What did Tom mean?

(29) Dick wrote:

$P$  is a false statement.

$$R = \{ \text{"Massachusetts is a city", "Connecticut is a city", "Missouri is a city", "Alaska is a city"} \}$$

What did Dick mean?

(30) Suppose I write:

I like  $P$ .

What would the truth set for this open sentence be for you?

(31) Kathy uses the symbol " $\sim$ " to mean "not." Kathy wrote:

$(\sim P)$  is false.

$$R = \{ \text{"The violin is a musical instrument", "The trumpet is a musical instrument", "The piano is a musical instrument", "The trombone is a musical instrument"} \}$$

What did Kathy mean?

(28) "St. Louis is a city" is a true statement.  
 "New York is a city" is a true statement.  
 "Los Angeles is a city" is a true statement.  
 "Miami Beach is a city" is a true statement.  
 "Minot is a city" is a true statement.

(29) "Massachusetts is a city" is a false statement.  
 "Connecticut is a city" is a false statement.  
 "Missouri is a city" is a false statement.  
 "Alaska is a city" is a false statement.

(30) This will be different for each student.

(31) "The violin is not a musical instrument" is false.  
 "The trumpet is not a musical instrument" is false.  
 "The piano is not a musical instrument" is false.  
 "The trombone is not a musical instrument" is false.

Notice that, because the English language does not work the same way that mathematical symbols do, the "not" symbol appears to the left of the  $P$  in mathematics, as

$\sim P$ ,

whereas the word *not* is usually inserted somewhere in the middle of the string of English words:

The violin is not a musical instrument.

Indeed, precisely where we insert the word *not* is, unfortunately, a matter of great importance. Notice that the statements

Not all men live in St. Louis

and

All men live not in St. Louis

have different meanings; indeed, the first by itself can have either of two different meanings:

It is not true that all the men in the world live in  $\overline{\text{St. Louis}}$ .  
 It is not true that only men live in St. Louis.

(32) Can you make up some examples like these?

(32) There are many possibilities.

(33) Is replacement set the same as truth set?

(33) No. The *replacement set* tells us which elements we have agreed to substitute for the variable. Not all of these substitutions will necessarily yield true statements. The *truth set* tells us which of these replacements agreed upon actually yield true statements.

It might be interesting here to introduce the word *subset*. A set  $A$  is called a subset of a set  $B$  if every element of  $A$  is also an element of  $B$ . This is written

$$A \subset B.$$

For example, if  $W = \{2, 4, 6\}$  and  $U = \{1, 2, 3, 4, 5, 6\}$ , then

$$W \subset U.$$

Using this word, we can say:

The *truth set* is a subset of the *replacement set*.

Notice that, for any set  $S$  whatsoever, it must always be true that

$$S \subset S.$$

There is one further quirk of mathematical language. Since the *empty set*,  $\emptyset$ , has no elements in it, then there are no elements of  $\emptyset$  that fail to be elements of  $S$ , for any set  $S$ . Hence, every element of  $\emptyset$  is an element of  $S$ , and the *empty set is regarded as a subset of every set*,

$$\emptyset \subset S.$$



## THE CARTESIAN PRODUCT OF TWO SETS

This chapter introduces two ideas: the Cartesian product of two sets and the idea of rectangular (or Cartesian) coordinates.

The formal definition of the Cartesian product of sets  $A$  and  $B$ , which we write

$$A \times B,$$

is " $A \times B$  is the set of all ordered pairs  $(x, y)$ , where  $x$  is an element of  $A$ , and  $y$  is an element of  $B$ ." Mathematicians sometimes use the symbol  $\in$  to mean "is an element of." Using this symbol, we could write

$$A \times B \text{ is the set of all ordered pairs } (x, y), \\ \text{where } x \in A, \text{ and } y \in B.$$



### CHAPTER 2

## The Cartesian Product of Two Sets

[page 5]

When we speak of an ordered pair of names (or numerals, or whatever), we mean a pair where order is important. Thus, the ordered pair

(Nancy, violin)

is not the same as the ordered pair

(violin, Nancy).

- (1) Suppose we write the open sentence

$P$  studies  $Q$ ,

and we agree to indicate replacements for the variables  $P$  and  $Q$  by writing:

The ordered pair  $(P, Q)$  may be either  
(Nancy, violin) or (Bill, clarinet).

What would we mean?

- (2) Joe says we would mean

violin studies Nancy

and

Bill studies clarinet.

Did Joe use the order correctly?

## ANSWERS AND COMMENTS

- (1) and (2) **Joe is wrong.** According to the agreement stated in the problem, the *first* element of each ordered pair is to be used as a replacement for the variable  $P$ , and the *second* element of each ordered pair is to be used as a replacement for the variable  $Q$ . We can write out the two uses of UV as follows:

Nancy  $\rightarrow P$

violin  $\rightarrow Q$

Result: Nancy studies violin.

Bill  $\rightarrow P$

clarinet  $\rightarrow Q$

Result: Bill studies clarinet.

If  $A$  is the set

$$A = \{Al, Bill, Henry\}$$

and if  $B$  is the set

$$B = \{Nancy, Eileen, Eva\},$$

then the Cartesian product  $A \times B$  is the set

$$\{(Al, Nancy), (Al, Eileen), (Al, Eva), \\ (Bill, Nancy), (Bill, Eileen), (Bill, Eva), \\ (Henry, Nancy), (Henry, Eileen), (Henry, Eva)\}.$$

(3) If

$$A = \{1, 2, 3\}$$

and

$$B = \{5, 6\},$$

can you write the Cartesian product

$$A \times B?$$

(4) Using the same sets as in question 3, can you write the Cartesian product

$$B \times A?$$

René Descartes was a French mathematician and philosopher who was born in 1596 and who died in 1650. He made very effective use of the idea of naming points in the plane by using ordered pairs. The adjective "Cartesian" is derived from Descartes' last name. As we shall see, discoveries made by Descartes in the seventeenth century continue to influence our lives today. [page 6]

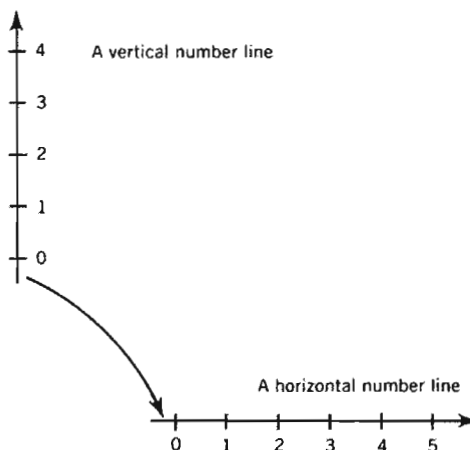
(5) Do you know how Descartes was able to use ordered pairs of numbers as names for points in the plane?

$$(3) \quad A \times B = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}$$

Notice that the number 1 *is not* an element of  $A \times B$ , even though (1, 5) and (1, 6) are elements of  $A \times B$ . Indeed, no *number* is an element of  $A \times B$ ; the elements of  $A \times B$  are *ordered pairs of numbers*. (Perhaps in an analogous way, a mother is not a family, although she may be *part* of a family.)

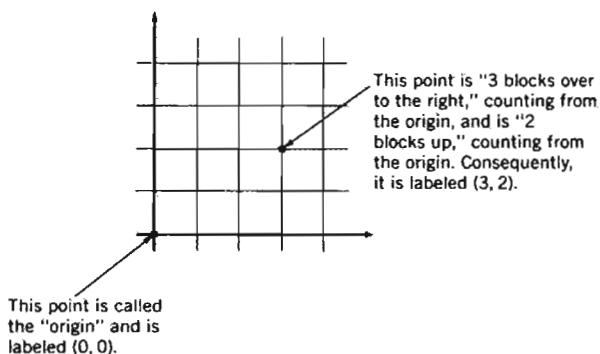
$$(4) \quad B \times A = \{(5, 1), (5, 2), (5, 3), (6, 1), (6, 2), (6, 3)\}$$

(5) Descartes' method was to "cross" two number lines:



"Cross" the two number lines so that the point labeled "0" on the vertical line becomes coincident with the point labeled "0" on the horizontal line.

By crossing the number lines Descartes obtained a grid (much as one uses streets and avenues in Manhattan):



(6) What was life in the United States like during Descartes' lifetime?

(6) **Descartes was born in 1596 and died in 1650. To compare this with life in the United States (which, of course, was not the "United States" in those days), one might consider, first of all, the dates displayed on the number line on page 33.**

We suggest that you cut out the number line on page 33, paste the sections together, and display it for your students' use during the discussion of question 6.

For useful references on life in the colonies during Descartes' lifetime, see Appendix A: Barck (148), Wright (162).

By contrast, life in France and Germany, where Descartes lived, showed presumably more adequate provision for intellectual endeavors, as witness the university founding dates, displayed on the number line on page 35. (Again, we suggest you cut out the number line, join the sections, and display it for your students.)

For references, consult Appendix A: Clark (150).

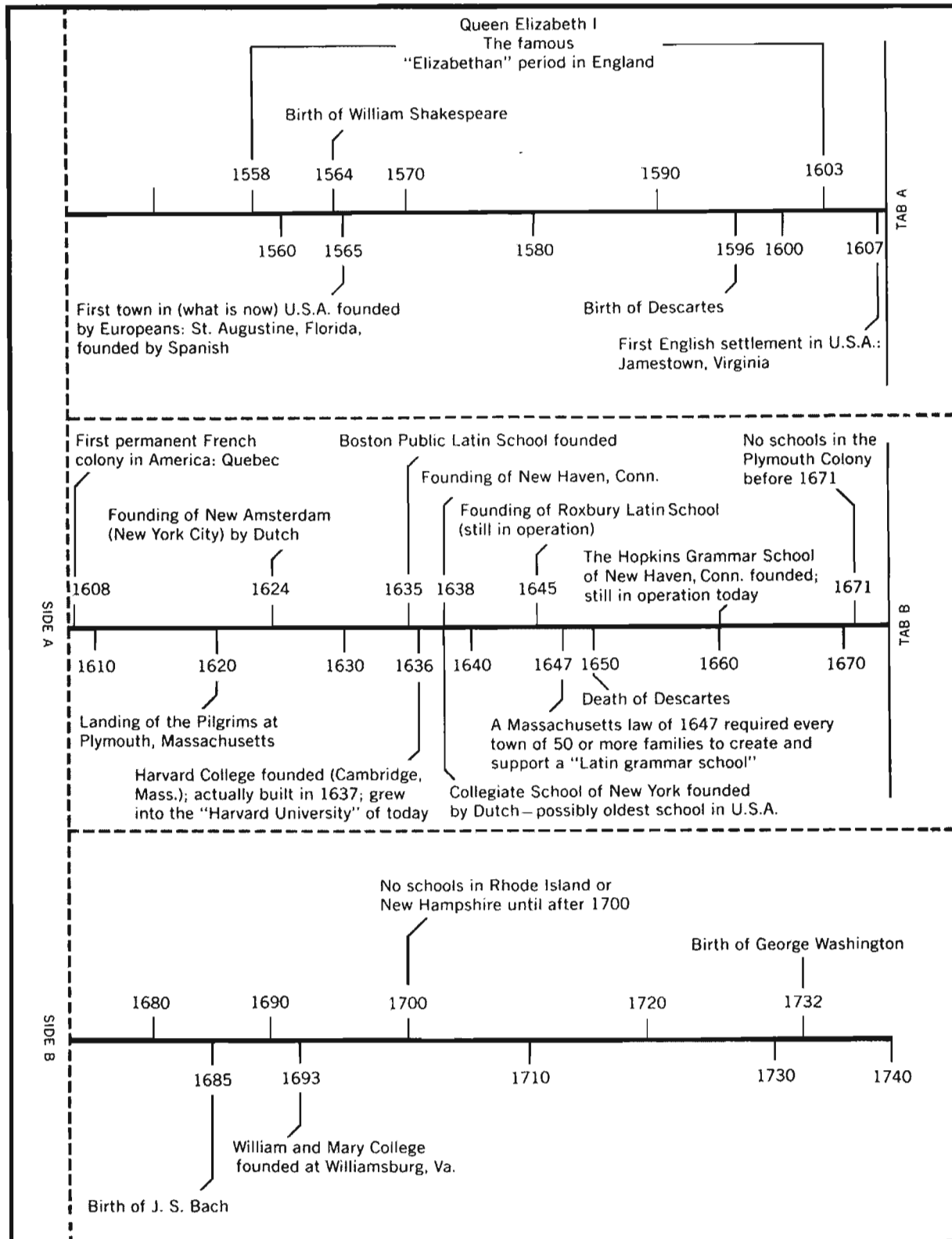
Further insight into the age when Descartes lived may be gleaned from the answers to questions 7 through 10.

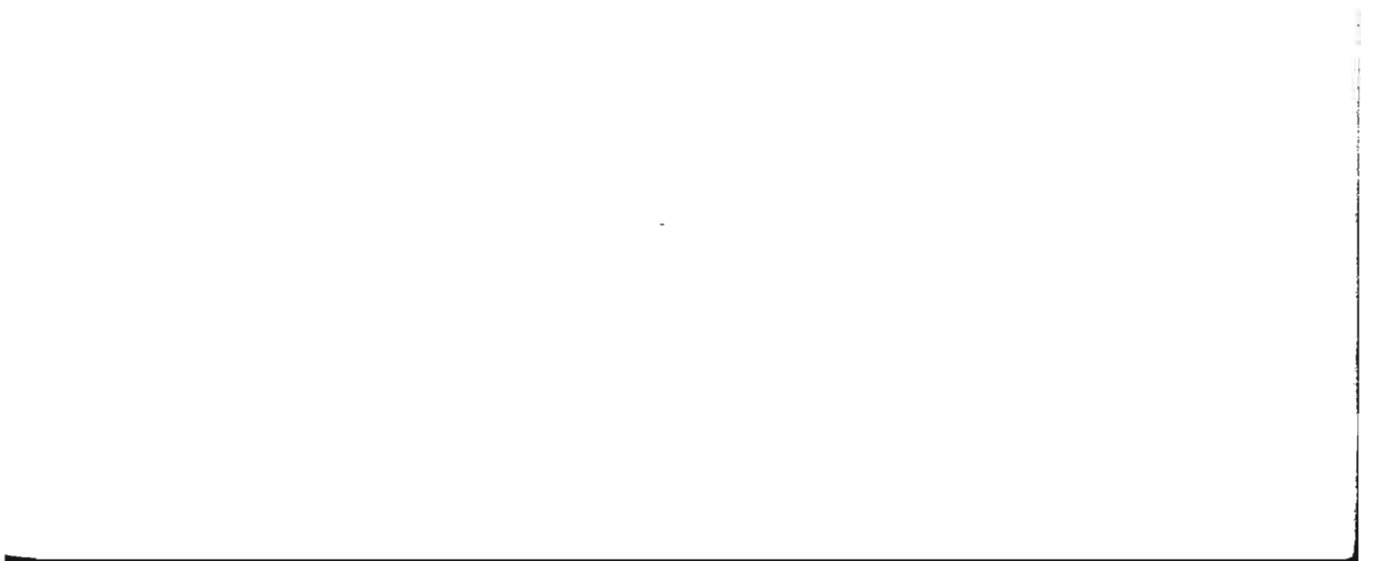
It may seem that Descartes lived a long time ago, but this is hardly the case. One can make convincing arguments that "our modern point of view" was born in Tudor England (among other places), and hence dates from around 1485. Certainly the music of Gabrielli and the elder Scarlotti still speaks meaningfully to us today—not to mention the plays of William Shakespeare.

That Descartes himself had much of a "modern" point of view may be illuminated by the following note which he wrote in his book *La Geometrie*:

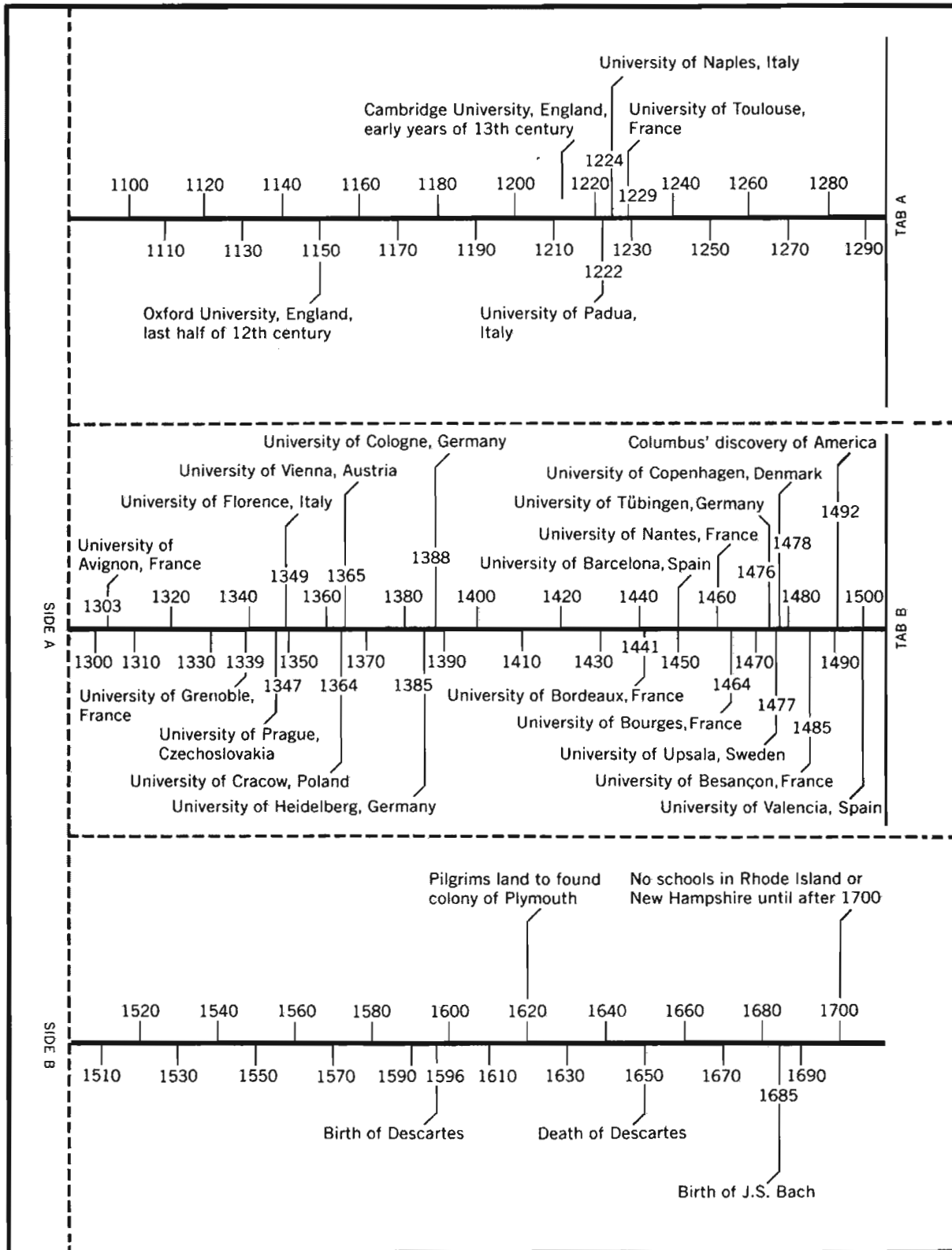
I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery.

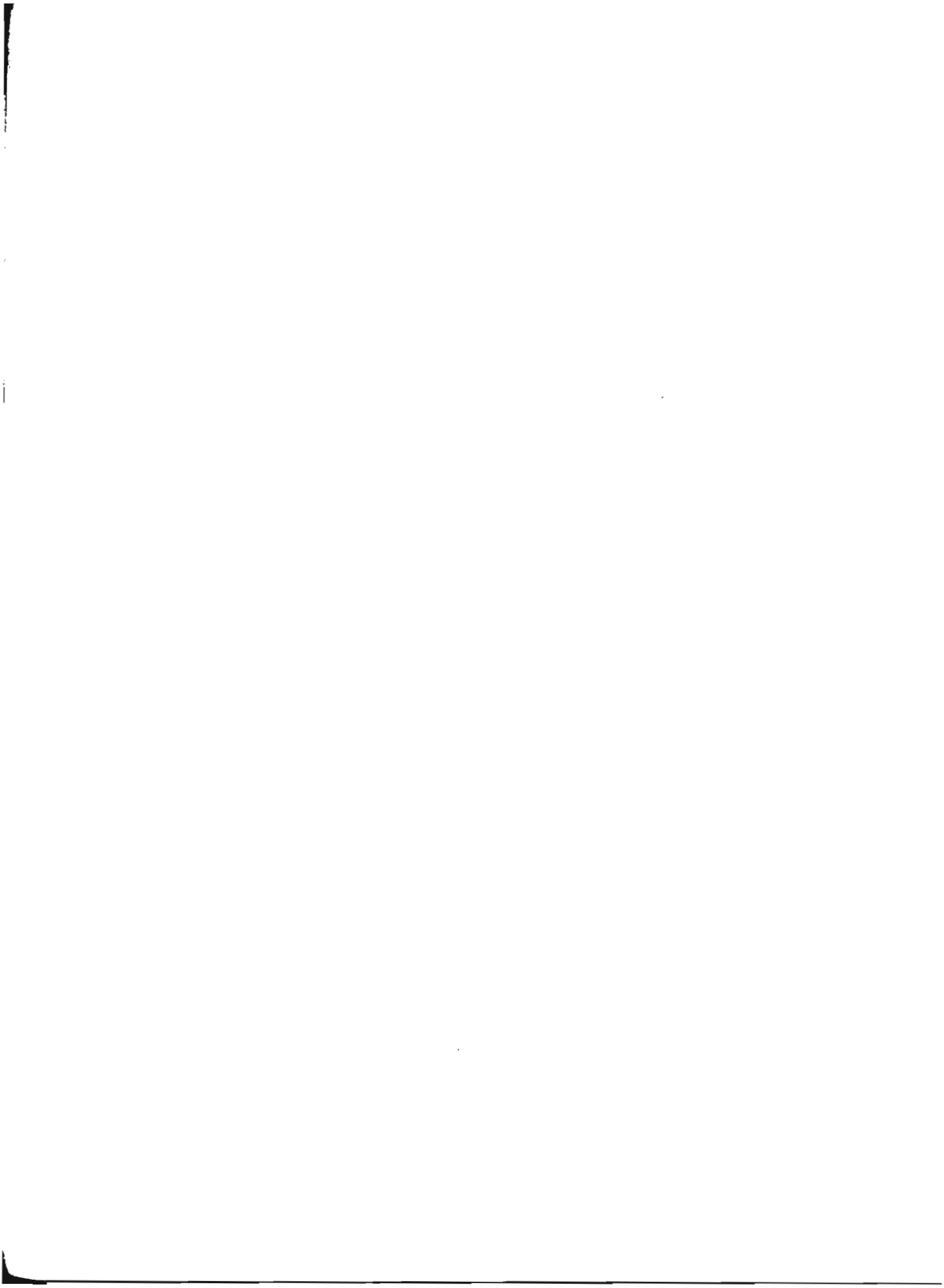
For further information on the life and times of Descartes, see Appendix A: Bell (149), Newman (158).

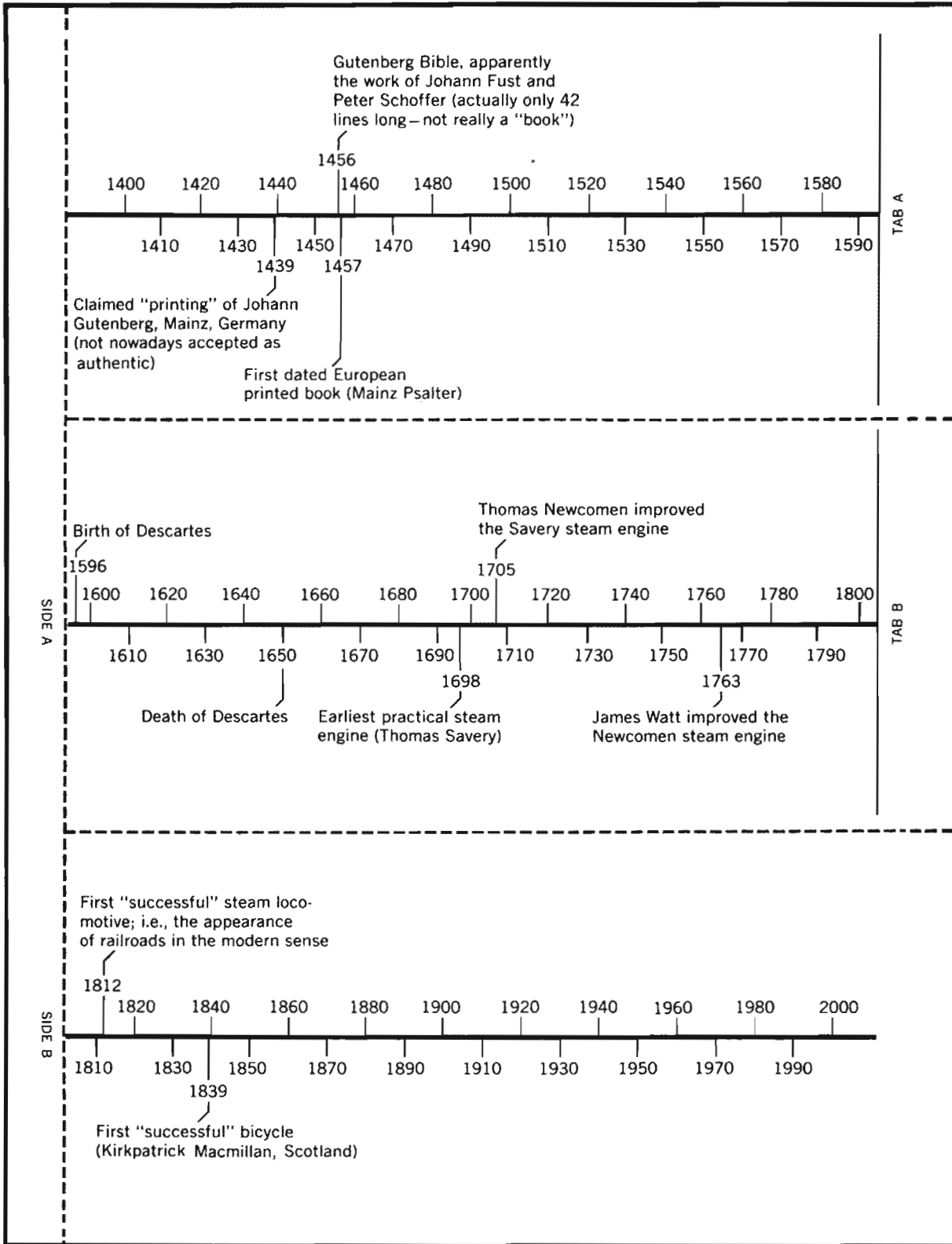














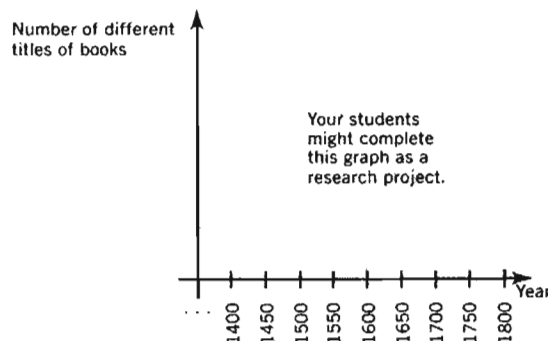
(7) During Descartes' lifetime did they have radios? television? telephone? airplanes? automobiles? bicycles? How did people travel in Descartes' time? Did they have railroads? steam engines? Did they have printed books in those days? Were they able to sail across the Atlantic Ocean?

(7) Obviously, they did not have radios, TV, telephones, airplanes, or automobiles. (Compare also the answer to question 6 above.)

The dates of a few important inventions and innovations are shown on the number line on page 37, which we suggest you cut out and display.

Obviously, this is the merest beginning of what can be done with number lines and graphs in attempts to illuminate the temporal ordering of history. You may find other number-line pictures more revealing than those presented here. Make some of your own if you like.

Furthermore, a graph, on rectangular (Cartesian) coordinates, that plots time in years on the vertical axis and the estimated number of different titles of printed books in existence (in Europe) at that time on the horizontal axis might be very revealing:



(8) What kind of music did they have in Descartes' time?

(8) Descartes, dying in 1650, never heard the music of Johann Sebastian Bach, who was born in 1685, although Bach is the earliest major Western composer whose works are well known at the present time. Interest in pre-Bach music nowadays runs high, and the time may not be far off when composers of Descartes' time will be more familiar than they are today.

An effective teaching unit might combine the major mathematical discovery of Descartes (the "crossed number lines," or Cartesian coordinates) with an attempt to get a glimpse of Western civilization in 1596 - 1650. The unit might include listening to some of the music of that period and discussing transportation in that period, life in the American colonies at that time, perhaps some literature of that period, etc.

The following are some recordings of music from the time of Descartes:

1. Erwin Bodky, playing harpsichord and clavichord, on the record entitled *Music of the Baroque Era for Harpsichord and Clavichord*, Unicorn Records, UN 1002. This recording includes music of Samuel Scheidt (1587 - 1654), one of the major pre-Bach composers, whose music has immediate appeal for the twentieth-century listener, and Matthias Weckmann (1619 - 1674).

2. *Recorder and Harpsichord Recital No. 3*, London label LL1026, which features Carl Dolmetch and Joseph Saxby playing the music of William Lawes (1602 - 1645) and also the famous folk tune, "Greensleeves," which was already "old" in Descartes' time.

3. Surely one of the major pre-Bach composers was Claudio Monteverdi. His "Madrigals, Book 1" (1587) can be heard on the Allegro LP recording ALG 3020, performed by the Roger Wagner Madrigal Singers.

4. A most significant area of music from this period is represented by Gregorian chants. An excellent recording is Period LP number SPL 569, *Gregorian Chants*, Vol. 1, performed by the Trappist Monks' Choir of Cistercian Abbey.

By way of contrast, you may want to listen to some music that was too "modern" for Descartes to have heard. For example, some Bach organ music performed on a Baroque organ (much is available); the Mozart "Sonatas for Violin and Harpsichord"; some harpsichord music of Domenico Scarlatti (1685 - 1757), for example, Fernando Valenti on Westminster LP XWN 18918; Vivaldi oboe and bassoon concertos; the harpsichord music of Pachelbel, Böhm, Rathgeber, and Fischer (available on the Bodky LP mentioned above); the symphonies and string quartets of Haydn (born in 1732); the symphonies of Beethoven; and the fifth symphony of Sergei Prokofiev. (Perhaps also, some contemporary "electronic" music.)

In connection with Descartes' use of coordinates, you may want to look at a text on analytic geometry and the study of conic sections, and to consult Appendix A: Bell (149) and Leason (155).

(9) Are there any schools or colleges in the United States today that were in existence during Descartes' lifetime? Are there any in England? in France? in Italy?

(9) In 1638 the Dutch in the New Amsterdam colony founded what is now known as the Collegiate School of New York, possibly the oldest private school in the United States. Roxbury Latin School, founded by John Eliot in 1645, still operates today, as does the Hopkins Grammar School (New Haven, Connecticut), which was founded in 1660. Boston Public Latin School was founded in 1635. Harvard College was founded in 1636 - 37, and William and Mary College was founded in 1693. The first universities in the "New World" were founded by the Spanish in the "New Spain" colonial empire. Many European universities had existed for centuries by the time Descartes was born. (Compare answers to questions 7 and 8.)

For further reading, see Appendix A: Hofstadter (153), Haskins (152).

(10) Did they have plays in Descartes' day? What books, novels, or plays, if any, might Descartes have read or seen?

(10) If you wish, you can use this question as a research assignment for some of your students.

(11) Jimmy wanted to give number names to points on a line. First, he named one point "0";



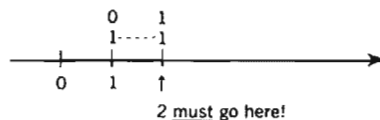
then he named a point "1."



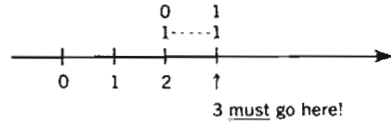
Can you give number names for some other points on Jimmy's line?



tells us, since  $1 + 1 = 2$ , that the segment  $\overline{01}$  "added" to itself must "equal" the segment  $\overline{02}$ :



Similarly,



One can continue this pattern indefinitely.

(12) Ellen used a vertical line.

She named one point "0."

Then she named another point "1."

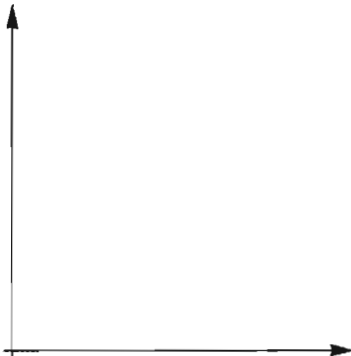


[page 7]

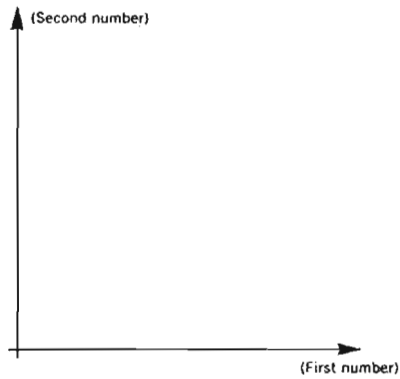
Can you give number names for some other points on Ellen's line?



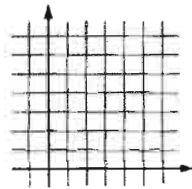
Descartes used a pair of crossed number lines in order to name the points in the plane.



Since he was going to use **ordered pairs of numbers**, he had to decide upon an order. He decided to use the **first number of the ordered pair to refer to the horizontal number line** and the **second number of the ordered pair to refer to the vertical number line**.

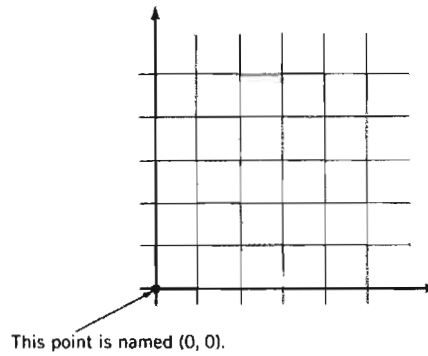


(13) If we draw a grid, like this,



can you find the point Descartes named  $(0, 0)$ ?

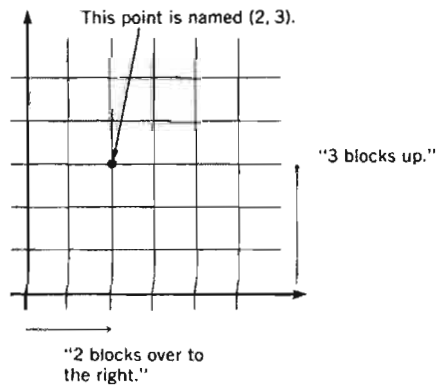
(13) The point  $(0,0)$ , which is often called the **origin**, is located where the vertical and horizontal axes (or number lines) cross:



[page 8]

(14) Can you find the point Descartes named  $(2, 3)$ ?

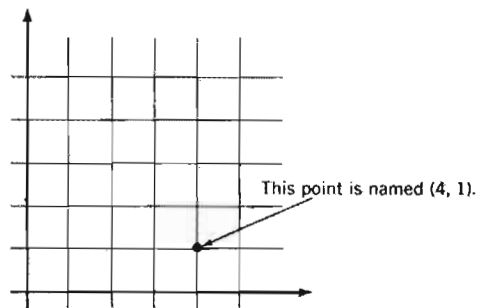
(14)



Notice that one counts "city blocks," rather than "intersections."

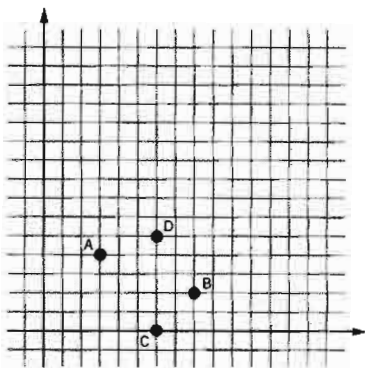


(15) Can you find the point Descartes named  $(4, 1)$ ? (15)



You may want to view the film "A Lesson with Second Graders."

(16) What names (using ordered pairs of numbers) would Descartes give to these points?



(16) **A: (3, 4)**

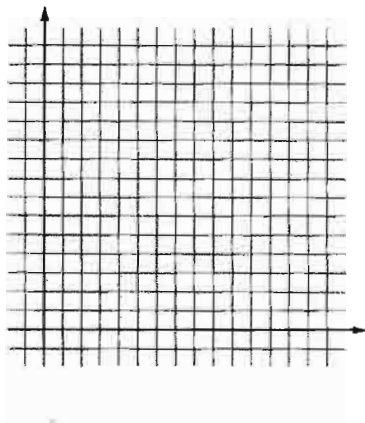
**B: (8, 2)**

**C: (6, 0)**

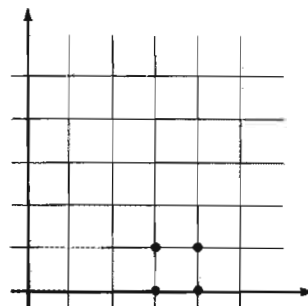
**D: (6, 5)**

(17) Descartes could use his "crossed number lines" to make pictures representing Cartesian products. Can you use these "Cartesian coordinates" to make a picture representing

$A \times B$ , if  $A = \{3, 4\}$  and  $B = \{0, 1\}$ ?

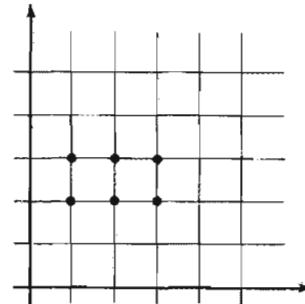


(17)  $A \times B = \{(3, 0), (3, 1), (4, 0), (4, 1)\}$



(18) Suppose  $A = \{1, 2, 3\}$ . Suppose  $B = \{2, 3\}$ . Can you make a graph (or picture) showing the Cartesian product  $A \times B$ ? How many points will there be in the picture representing  $A \times B$ ?

$$(18) A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$$



There are 6 points in this picture of  $A \times B$ .

(19) Suppose that the set  $M$  has  $r$  elements and the set  $N$  has  $s$  elements. How many points will there be in the picture representing the Cartesian product

$$M \times N?$$

(19) There will be  $r \times s$  points.

(20) Suppose that  $A = \{\text{Nancy, Jane}\}$  and  $B = \{\text{Don, Roy, Louis}\}$ . Can you write the Cartesian product

$$A \times B?$$

$$(20) A \times B = \{(\text{Nancy, Don}), (\text{Nancy, Roy}), (\text{Nancy, Louis}), (\text{Jane, Don}), (\text{Jane, Roy}), (\text{Jane, Louis})\}$$

[page 9]

(21) Suppose that  $A = \{\text{red, green, yellow}\}$  and  $B = \{\text{hat, scarf}\}$ . Can you write the Cartesian product

$$A \times B?$$

$$(21) A \times B = \{(\text{red, hat}), (\text{red, scarf}), (\text{green, hat}), (\text{green, scarf}), (\text{yellow, hat}), (\text{yellow, scarf})\}$$

(22) Suppose that  $A = \{2, 3, 4\}$  and  $B = \{1, 2\}$ . Make one graph to show  $A \times B$  and another graph to show  $B \times A$ .

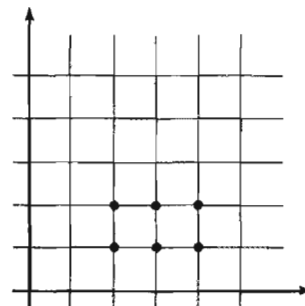
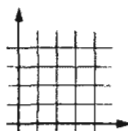
$$(22) A \times B = \{(2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$$

We can play a game using Descartes' method of naming points in the plane by means of ordered pairs. This game is just like tic-tac-toe, only different.

In this game, called "four-in-a-row," if you get four of your marks in an uninterrupted straight line, you win. All of your marks, unlike in tic-tac-toe, will be on the intersection of two lines. (The board allows room for five marks in a line.)

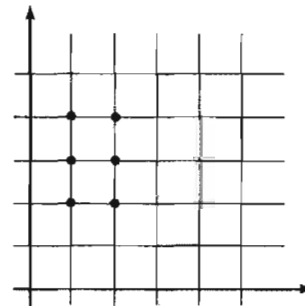
One team's marks are  $\times$ 's, and the other team's marks are  $\circ$ 's. The teams take turns naming points they want marked, using Descartes' system of ordered pairs of numbers. The teacher marks the points that each team names. (If you make an illegal move, you lose that turn, and no point is marked.)

The "playing board" looks like this:



Graph of  $A \times B$

$$B \times A = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$$



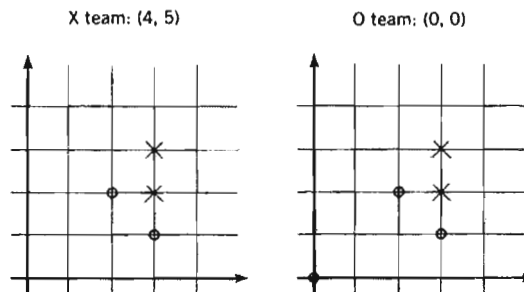
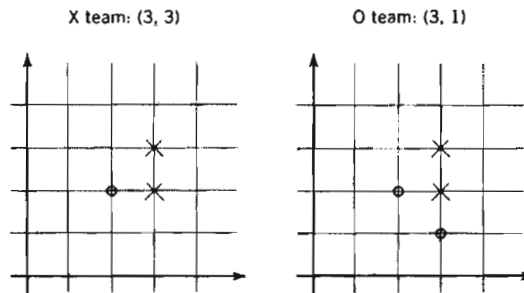
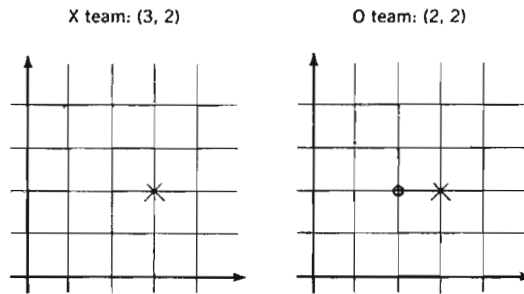
Graph of  $B \times A$

(23) Here is a sample game. See if you can keep track of it.

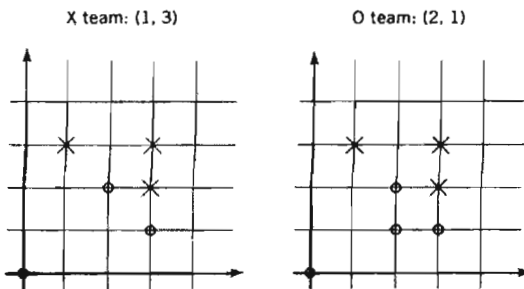
- x team: (3, 2)
- o team: (2, 2)
- x team: (3, 3)
- o team: (3, 1)
- x team: (4, 5)
- o team: (0, 0)
- x team: (1, 3)
- o team: (2, 1)

What does the board look like now?

(23) Here is the game, step by step:



Since (4, 5) was illegal, no point was marked.



(24) Can you finish the game that we have just started? Which team do you think was ahead at the end of question 23?

(24) **The x team will win if they don't make any mistakes.**

## OPEN SENTENCES WITH MORE THAN ONE VARIABLE

In this chapter we want to consider open sentences such as

$$\begin{aligned}\square + \triangle &= 8 \\ (1 \times \square) + 3 &= \triangle \\ (\square \times \square) + (\triangle \times \triangle) &= 25\end{aligned}$$

and so on. We shall, in most cases, be dealing with two variables,  $\square$  and  $\triangle$ . In such cases, the "rule for substituting" says that:

Whatever number we put in one  $\square$  must be put in all the other  $\square$ 's; whatever number we put in one  $\triangle$  must be put in all the other  $\triangle$ 's; the number in the  $\triangle$  may be the same as the number in the  $\square$  or it may be different—we get our choice!

Thus, for the open sentence

$$(\square + \square) + \triangle = (\square + \triangle) + \square,$$

both of the following substitutions are legal, according to the rule for substituting:

$$\begin{aligned}(\underline{3} + \underline{3}) + \underline{3} &= (\underline{3} + \underline{3}) + \underline{3} \\ (\underline{3} + \underline{3}) + \underline{4} &= (\underline{3} + \underline{4}) + \underline{3}\end{aligned}$$

With the two variables  $\square$  and  $\triangle$ , there are two common methods of indicating the *replacement sets*:

(i) We may use subscripts, as here, so that

$$\begin{aligned}\square + \triangle &= \triangle + \square \\ R_{\square} &= \{1, 2\} \\ R_{\triangle} &= \{5, 6\}\end{aligned}$$

means exactly the same as this list:

$$\begin{aligned}1 + 5 &= 5 + 1 \\ 1 + 6 &= 6 + 1 \\ 2 + 5 &= 5 + 2 \\ 2 + 6 &= 6 + 2\end{aligned}$$

(ii) We may use *ordered pairs* of numbers. In this case, we agree that the first number in the ordered pair is the replacement for the variable  $\square$  and that the second number in the ordered pair is the replacement for the variable  $\triangle$ . Thus,

$$\begin{aligned}\square + \triangle &= \triangle + \square \\ R &= \{(1, 1), (1, 3), (2, 7)\}\end{aligned}$$

means exactly the same as this list:

$1 + 1 = 1 + 1$
$1 + 3 = 3 + 1$
$2 + 7 = 7 + 2$

Notice that the second method allows greater flexibility. Indeed, one could not have written the second example by the first method.



## CHAPTER 3

## Open Sentences With More Than One Variable

[page 10]

(1) Sometimes we want to write an open sentence that has more than one variable in it. Do you know how we can do it? Can you give some examples?

(2) Nancy gave these examples:

$$\square + \triangle = 8$$

$$\square + 3 = \triangle$$

$$\square + \square = \triangle$$

$$\square + \square + \triangle = 9$$

Can you figure out what Nancy meant? How would the "rule for substituting" work for Nancy's open sentences?

(3) Bill says that Nancy forgot to indicate any replacement sets for her variables. What do you think?

(1) **See the preceding remarks.**

(2) **See the preceding remarks.**

(3) **Bill is, of course, correct. When we don't bother to indicate any replacement sets, we usually mean "use any numbers that you know." (Sometimes we mean "use the counting numbers 1, 2, 3, ...")**

There is an important matter here which deserves attention. Some authors nowadays use the phrase "counting numbers" to mean the set  $\{0, 1, 2, 3, \dots\}$ , while other writers use these same words to refer to the set  $\{1, 2, 3, \dots\}$ . We would argue that, in the interests of simplifying the over-all elementary school program, we want to focus attention on the *major* number systems used in elementary school, and we wish to keep this number as small as possible. A modern elementary school mathematics program cannot be built with less than three distinct number systems, which we have come to refer to as

the system of *counting numbers*  $\{0, 1, 2, 3, \dots\}$ ,

the system of *reference-point numbers*  $\{\dots, -2, -1, 0, +1, +2, \dots\}$ ,

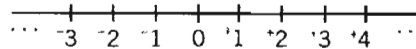
the system of *measurement or "sharing" numbers*  $\{\frac{1}{2}, \frac{-1}{2}, \frac{2}{3}, 1, 1.9, 0.01, \dots\}$ .



## ANSWERS AND COMMENTS

We will be studying all three of these systems as we work our way through this book. Please don't worry if they are unfamiliar right now.

The point here is that, if we want to simplify the over-all elementary school program as much as possible, these three systems seem to be the really basic ones. The "counting numbers" answer the commonplace question "how many?" The reference-point numbers occur in any situation where we mark an arbitrary reference point (like the "zero" on a thermometer) and can move away from this reference point in either of two directions (as "toward higher temperatures" or "toward colder temperatures"). The geometric picture for this is:



The "sharing" or "measurement" numbers occur when we share things or measure things (as, " $\frac{1}{2}$  of the candy" or "the table is two and a half feet high").

If we agree to focus on these three number systems as the major ones for modern elementary school mathematics, then we are forced to include zero among the set of "counting numbers," since zero is often the answer to the question "how many?" (For example, in "How many sisters do you have?" or "How many children do you have?" and so on.)

*In the examples in this chapter, we shall be using the system of counting numbers.* Later on in this book we shall consider problems just like these, and approach them using one of the other two number systems. This step-by-step approach is simpler, both for reader and author, in a textbook situation. In an actual classroom situation, where we enjoy the greater power of two-way communication, we would move into other number systems *when the learner began to indicate a readiness to do so*—which might come very early in the year's work.

(4) In Nancy's class, it turned out that some students knew about "negative numbers" and other students did not. In order to be fair, the class agreed to work on Nancy's open sentences, **without using any negative numbers**. Then, in order to make the work easier, the class also agreed **not to use any fractions**. In order to remember this agreement, the class wrote:

$$R_0 = \{0, 1, 2, 3, \dots\}$$

$$R_\Delta = \{0, 1, 2, 3, \dots\}$$

What do you think the class meant?

(5) If

$$R_{\square} = \{0, 1, 2, 3, 4, \dots\}$$

$$R_\Delta = \{0, 1, 2, 3, 4, \dots\},$$

can you find the truth set for each of Nancy's open sentences?

(4) **The class was using the first method described above.**

(5) (a)  $\square + \Delta = 8$

$\square$	$\Delta$
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0

Table for the Truth Set

An alternative way to write the truth set would be:

$$T = \{(0, 8), (1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 0)\}.$$

Notice that  $T$  is a set of ordered pairs of numbers. It is *not* a set of numbers. For example, 0 is not an element of  $T$ , even though (0, 8) and (8, 0) are both elements of  $T$ . Similarly, *far* is a word in the English language, but *f* is *not* a word in the English language.

As a matter of fact,  $T$  is a subset of the Cartesian product

$$R_0 \times R_0.$$

We can write this, using the symbol  $\subset$  (which means "is a subset of"), as follows:

$$T \subset R_0 \times R_0.$$

- (b)  $\square + 3 = \triangle$ . The truth set in this case is an infinite set, so we shall have to use the "three-dots" notation to mean "and it goes on like this forever and never comes to an end."

$\square$	$\triangle$
0	3
1	4
2	5
3	6
⋮	⋮

Table for Truth Set

Alternatively,

$$\tau = \{(0, 3), (1, 4), (2, 5), (3, 6), \dots\}.$$

- (c)  $\square + \square = \triangle$

$\square$	$\triangle$
0	0
1	2
2	4
3	6
⋮	⋮

Table for Truth Set

Alternatively,

$$\tau = \{(0, 0), (1, 2), (2, 4), (3, 6), \dots\}.$$

- (d)  $\square + \square + \triangle = 9$

$\square$	$\triangle$
0	9
1	7
2	5
3	3
4	1

Table for Truth Set

There are only 5 elements in this truth set:

$$\tau = \{(0, 9), (1, 7), (2, 5), (3, 3), (4, 1)\}.$$

(6) Bruce said he could write the truth set for the open sentence

$$\square + \triangle = 8$$

by means of a table, like this:

$\square$	$\triangle$
0	8
1	7
2	6
3	5

Table for Truth Set

What do you think?

[page 11]

(7) Ellen says that Bruce's table is not complete. Can you finish it?

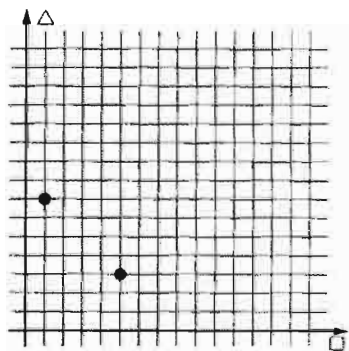
(8) Can you use a table to represent the truth sets for Nancy's other open sentences (see question 2)?

(9) Can you make up some open sentences of your own?

(10) Jerry says he can use Descartes' idea of "crossed number lines" to represent the truth set for the open sentence

$$\square + \triangle = 8.$$

by means of a graph. He labeled the horizontal number line with a  $\square$  to show that he used it to locate the replacement for  $\square$ . He labeled the vertical number line with a  $\triangle$  to show that he used it to locate the replacements he used for  $\triangle$ .



Graph for Truth Set

(6) Bruce's table is not complete. See the answer to question 5(a).

(7) Ellen is correct. See the answer to question 5.

(8) See the answer to question 5.

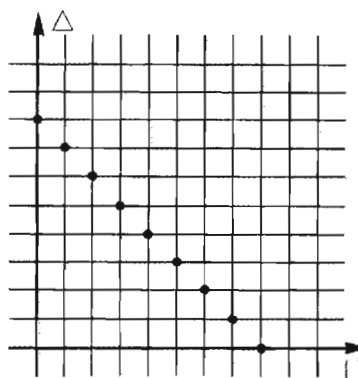
Notice that questions 5, 6, 7, and 8 present a fairly typical "Madison Project" sequence, as follows:

- (i) General question
- (ii) Question to help with (i)
- (iii) Further question to help with (i)
- (iv) Repeat of question (i)

(9) This should give no difficulty.

(10) The complete graph looks like this (recalling that

$$R_{\square} = R_{\triangle} = \{0, 1, 2, 3, \dots\}):$$



What would this graph look like if  $R_{\square} = \{0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \dots\}$  and  $R_{\triangle} = \{0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \dots\}$ ? (You may wish to view the film "Graphs and Truth Sets.")



Jerry says the points marked correspond to this table:

$\square$	$\triangle$
1	7
5	3

Can you complete the graph, if

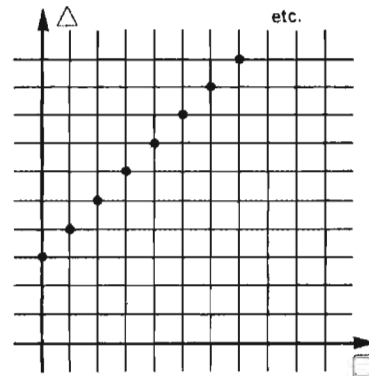
$$R_{\square} = \{0, 1, 2, 3, 4, 5, \dots\}$$

$$R_{\triangle} = \{0, 1, 2, 3, 4, 5, \dots\}?$$

Can you make a graph to show the truth set for each of the following equations? (Use  $R_{\square} = \{0, 1, 2, 3, \dots\}$ ,  $R_{\triangle} = \{0, 1, 2, 3, \dots\}$ .)

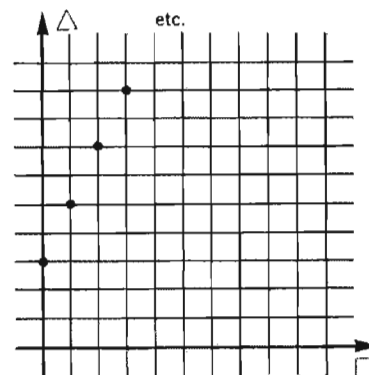
(11)  $(\square \times 1) + 3 = \triangle$

(11)



(12)  $(\square \times 2) + 3 = \triangle$

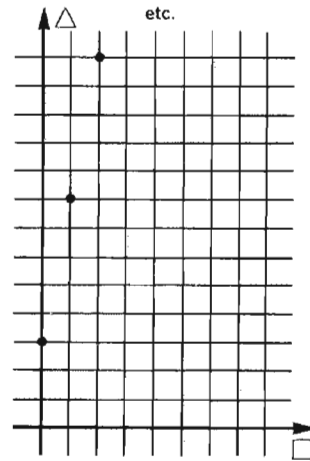
(12)



You may want to view the film "Second Lesson" and to compare Chapter 11 of *Discovery*.

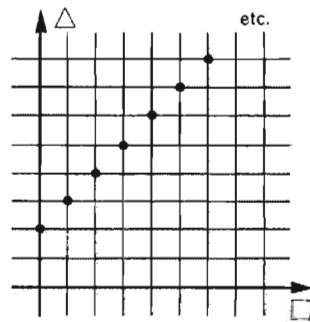
(13)  $(\square \times 5) + 3 = \triangle$

(13)



(14)  $(\square \times 1) + 2 = \triangle$

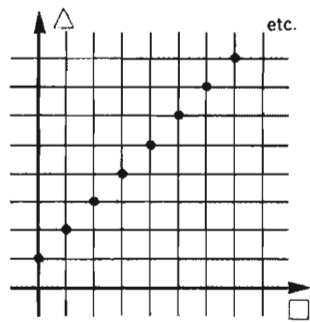
(14)



See Chapter 15 of *Discovery*.

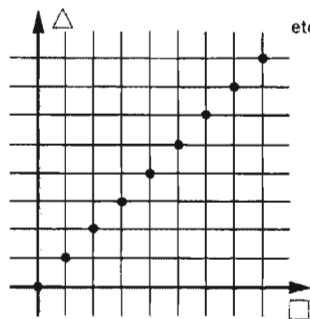
(15)  $(\square \times 1) + 1 = \triangle$

(15)



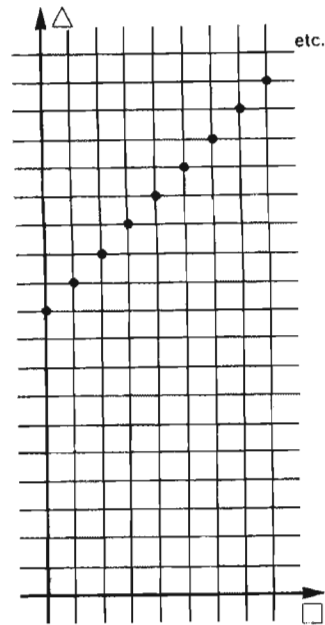
(16)  $(\square \times 1) + 0 = \triangle$

(16)



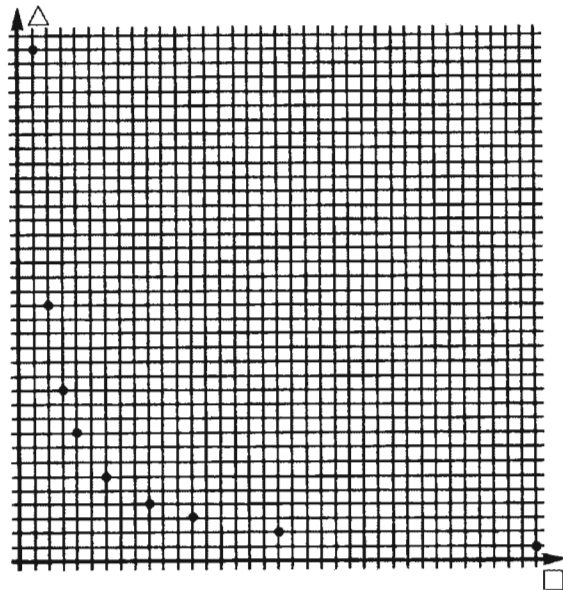
(17)  $(\square \times 1) + 10 = \triangle$

(17)



(18)  $\square \times \triangle = 36$

(18) Recall that  $R_\square = \{0, 1, 2, 3, \dots\}$  and  $R_\triangle = \{0, 1, 2, 3, \dots\}$ . The graph to represent the truth set of  $\square \times \triangle = 36$  is then as follows:



Evidently, then,  $T$  contains nine elements, each of which is an ordered pair of numbers.  $T = \{(1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1)\}$ .

## SIGNED NUMBERS

Your students may already know about signed numbers. If they do, this chapter will provide a good opportunity to see that their understanding is really adequate for future work. If they don't—and we are operating under the assumption that they do not—then this chapter is intended to be the first introduction of the idea of negative numbers.

There are several remarks about this chapter that may clarify its contents:

(i) The official “first real introduction” of signed numbers is presented by means of the “pebbles-in-the-bag” model. This model permits one to add and subtract unsigned numbers, and to express the answer as a signed number, as in

$$\begin{aligned}5 - 3 &= +2 \\5 - 9 &= -4.\end{aligned}$$

It does *not* provide for the addition or subtraction of *signed numbers*; for example, it does not provide for

$$\begin{aligned}+3 + +2 \\+3 - -5.\end{aligned}$$

In order to handle problems in adding and subtracting signed numbers, we shall use a model involving “postman stories,” in Chapter 5.

(ii) In the present chapter, we shall distinguish *counting* situations (where  $5 - 3 = 2$  and where  $3 - 5$  is impossible) from *reference point* situations (where  $5 - 3 = +2$  and  $3 - 5 = -2$ ).

This distinction should become easy with a little experience. (In our own work, we have never had trouble with this distinction, but teachers are sometimes apprehensive when they first encounter it.)

(iii) We shall use a “modern” notation, pioneered by Max Beberman and the UICSM Project, and write  $+2$ ,  $-3$  instead of  $+2$ ,  $-3$ .

We shall read  $+2$  as “positive two,” and shall avoid reading it as “plus two.” Similarly,  $-2$  will be read as “negative two” and not as “minus two.” This language may seem strange at first; however, within our experience it clarifies matters considerably. This will be explained at some length later in this chapter.

(iv) Prior to the “pebbles-in-the-bag” model, we shall consider—very briefly indeed—a “hotel” model. We do not do anything serious with this model; indeed, you could omit it if you prefer. Its inclusion is merely a matter of building readiness for what follows.

For further reading, you may want to refer to Appendix A: Levi (90), Gibb (37), and Moise (95).



## CHAPTER 4

## Signed Numbers

[page 12]

The numbers 1, 2, 3, 4, ... arise whenever we count things. Really, these are the only numbers that arise from counting, at least the way most people do it.

If we use our imaginations, we might think to add zero, which arises in counting how many brothers you have (if you don't have any brothers).

This gives us

$$0, 1, 2, 3, 4, 5, \dots$$

(1) Can you mark 0, 1, 2, 3, 4, ... on a number line?

When we want to divide things up (like cakes and pies and candy bars) or when we want to measure things, we need more numbers, such as

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{10}, \frac{2}{10}, 2\frac{1}{2}, 2.7, \text{ and so forth.}$$

(2) Can you show the numbers

$$\frac{1}{2}, \frac{1}{3}, 1\frac{1}{4}, 2\frac{3}{10}, 4\frac{1}{2}, 3.7$$

on a number line? Where would  $3\frac{1}{100}$  be?

(3) Do you think there are any new kinds of numbers that are different from

counting numbers: 1, 2, 3, 4, ...

and different from

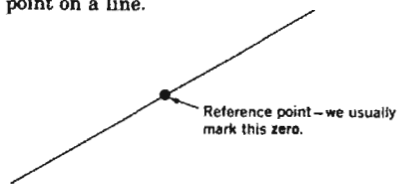
zero: 0

and different from

fractions:  $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, 1\frac{1}{10}, \frac{22}{100}$ , and so forth?

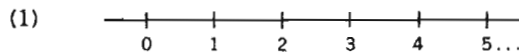
Do you know any other kinds of numbers?

Mathematicians talk about something that they call a one-dimensional linear vector space. You meet this one-dimensional linear vector space when you mark a reference point on a line.



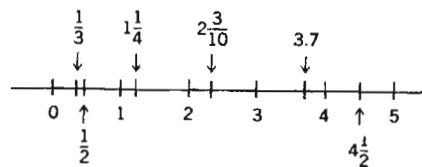
You can move away from this reference point in one direction or in the opposite direction.

## ANSWERS AND COMMENTS



Actually, even merely marking these points on a number line may serve to raise the question, "What happens when you go toward the left?" (You may be interested in viewing the films "A Lesson with Second Graders," and "Education Report: The New Math.")

(2) This will presumably give no trouble. Here are a few of the points in question.



(3) We hope the students may suggest "negative" numbers or "below zero" numbers. Sophisticated students may also know irrational numbers and complex numbers (which they may know in matrix form).

(4) Can you think of anywhere that you have seen numbers used this way?

(5) Do you know how they determine zero on a centigrade thermometer?

A certain hotel is built on the side of a steep hill. The result is that the entrance and the lobby are really in the middle of the building. The architect, who was also an amateur mathematician in his spare time, decided to label the lobby floor zero. The next floor above the lobby floor he called **positive one**. On the elevator indicator he wrote [page 13]

+1.

The floor just below the lobby level he called **negative one**. On the elevator indicator he wrote

-1.



- (4) Here are some possibilities: The scale on a thermometer. The scale on ammeters, especially old-fashioned automobile ammeters (which often recorded "discharge" and "charge" along a portion of a "number line," from -20 to +20. Altitude, above and below sea level. The number line. The axes in Cartesian coordinates (which is really the same as the number line).
- (5) Zero on a centigrade thermometer is determined as the equilibrium point of ice and water, at a controlled barometric pressure (or altitude).

We want to suggest that you use the symbols +3, -5, etc., rather than the traditional +3, -5, and that you read them as "positive three" and "negative five" instead of the traditional "plus three" and "minus five." The traditional notation gave three different meanings to - (read "minus") and two different meanings to + (read "plus"). Our work with children has convinced us that *this causes confusion*\*. Consequently, for the different meanings we use different symbols, as follows:

Written	Read	Meaning
+2	"positive two"	for <i>positive</i> numbers
2 + 3 or +2 + 5	"plus"	for the <i>operation</i> of addition
-3	"negative three"	for <i>negative</i> numbers
5 - 2 or -3 - 5	"minus"	for the (binary) operation of subtraction
°(-2)	"opposite" or "additive inverse"	for the (unary) operation of taking the additive inverse (cf. page 77)

The "modern" notation for positive, negative, and additive inverse was pioneered by Max Beberman and the UICSM Project; it is coming to be found in an increasing number of "modern" books.

\*For example, using traditional notation, a seventh-grade class was asked to make up an arithmetic of signed numbers and to defend it in terms of its consistency with previous mathematical work. They defined  $(+2) \times (-3)$  to be 3, on the grounds that "2 times 3 is 6, then you have to subtract 3, so the answer is 3." This might be called a rational response to an irrational notation; it confuses operations with part of the name of a number. Such confusion has never arisen when we ask children to decide what  $+2 \times -3$  should be; the operation here is clearly multiplication, while the - sign is clearly part of the name of the number "negative three."

To give a second example, "bright" engineering students in college used to be confused by the assertion that the absolute value of  $x$  was defined as:

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

They would ask, "How can the absolute value of  $x$  ever be minus?"

This, again, shows confusion as to what - means. When, instead, we have used the "modern" notation  $^{\circ}\square$  instead of  $-x$ , and asked fifth graders whether  $^{\circ}\square$  is positive or negative, they have refused our invitation to confusion, and have (correctly) insisted that it depends upon what you put in the  $\square$ .

- (6) Suppose the elevator starts at

-3

and goes to the lobby. Did it go up or down? How many floors up or down did it go?

- (7) Suppose the elevator starts at

+8

and goes to the lobby. Did it go up or down? How many floors up or down did it go?

- (8) Suppose the elevator starts at

-3

and goes to

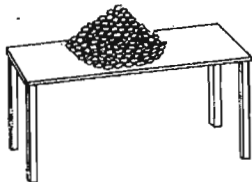
+6.

How far did it go? Which way, up or down?

- (9) Danny has a bag that has lots of pebbles in it.



On the table there is a pile with lots more pebbles.



[page 14]

We need to mark a reference point, so Jerry says "Go!" Now Danny takes 3 pebbles from the pile on the table, and puts them in the bag.

Are there more pebbles in the bag than there were when Jerry said "Go," or are there less? How many more or how many less? Do you know how we write this?

- (6) It went up; in fact, it went up three floors.

- (7) It went down; in fact, it went down 8 floors.

- (8) It went up 9 floors.

Questions such as 6, 7, and 8 are part of the standard Madison Project practice of trying to shape concepts in a preconscious (or preverbal) form, well *before* we try to elevate the concept to the conscious, explicit, verbal level. If you expect concepts to "come from nowhere," you will be disappointed. Nor can you build concepts at the verbal level. There must be previous preverbal or preconscious experience that has "built readiness"—i.e., already begun to create the concept at a preconscious level. As one example, E. E. Moise of Harvard University has pointed out that a child's experience of seeing rows of "identical" items on display in stores helps build readiness for the geometrical concept of *congruence*. As a further example, the work in *Discovery* on "balance pictures" is intended to help form, at a preconscious level, the concepts of *equivalent equations* and *transform operations*.

- (9) There are more pebbles in the bag; in fact, there are three more, which we write as +3.

You can watch this "pebbles-in-the-bag" procedure in complete detail, as a first introduction of signed numbers, in the film entitled "A Lesson with Second Graders."

This lesson always goes more smoothly if you ask "more or less" *before* you ask "how many more" or "how many less."

The question "Do you know how to write this?" is partly rhetorical: there is no (official) way that the child could know. Still, we get his attention better by asking questions. (What's more, it may even turn out that he *does* know!) The answer, of course, is that we write this as +3, which we read as "positive three."

It is important here to distinguish what we put into the bag (or take out) versus the signed number which represents the condition of the bag (i.e., more or less pebbles than when Jerry said "Go!"). What we put in we *count*; it is indicated by an *unsigned counting number*. What we take out we *count*; it, too, is represented by an *unsigned counting number*. However, when we ask for a report on the present condition of the bag—i.e., "Are there more or less stones than when Jerry said 'Go!?' How many more?"—we are *asking for a signed number*.

In the present case, our procedure is always as follows:

When Danny puts three pebbles into the bag, *this is counting*, and the teacher writes on the board

3.

(10) Now Danny takes 5 pebbles out of the bag. We can write

$$3 - 5.$$

Are there more pebbles in the bag than there were when Jerry said "Go," or are there less? How many more or how many less? Do you know how to write this

$$3 - 5 = \underline{\quad} ?$$

When the teacher asks, "Are there more or less than when Jerry said 'Go!'" this is *not counting*, and the teacher writes

$$3 = +3.$$

After writing this, which was intended to introduce the notation "-3," the teacher erases part of it, to end up with

$$3.$$

(10) **There are now two pebbles less in the bag than there were when Jerry said "Go!" We write**

$$3 - 5 = -2,$$

which we read as

**"three minus five equals negative two."**

In question 10, Danny removes five pebbles. *This, too, is counting*, and calls for *unsigned numbers*:

$$3 - 5.$$

When we get a report on the condition of the bag ("Two pebbles less than when Jerry said 'Go!'"), *this is not a matter of counting*, and calls for a *signed number*:

$$3 - 5 = -2.$$

Thus, the pebbles-in-the-bag model permits us to add and subtract unsigned numbers, and to express the answer as a signed number.

In many years of trials, we have consistently found the "pebbles-in-the-bag" model to be simple and effective. It is, however, important to teach it so as to avoid various possible pitfalls. Consequently, the Madison Project has developed a large battery of teacher aids that focus on the introduction of signed numbers, via the "pebbles-in-the-bag" model.

Most important of these is the film entitled "A Lesson with Second Graders." We strongly suggest that you view this film with a group of colleagues, and that thereafter you practice teaching the "pebbles-in-the-bag" model while your colleagues observe and make suggestions. With some cooperative effort of this sort, you can quickly learn the most effective methods for using this presentation with children in grades 2 through 9, or so.

Another important teacher aid is the Madison Project *In-Service Course #1 For Teachers*.

We can summarize a few important aspects of the "pebbles-in-the-bag" model briefly:

(i) Notice that we do *not* ask about the *total number of pebbles in the bag*. To do so would, of course, get us back into a *counting* situation where  $5 - 3 = 2$  and where  $3 - 5$  is impossible.

(ii) Instead, we have a child mark a *reference point* by saying "Go!," and our questions are always about more or less pebbles in the bag than there were when the child said "Go!"

This gets us out of "counting" situations and into a *reference point* situation, where

$$\begin{aligned} 5 - 3 &= +2 \\ 3 - 5 &= -2. \end{aligned}$$



(iii) Notice that we must begin with a large unknown number of pebbles already in the bag, and a pile of pebbles not in the bag, in order to enable us to "go either up or down" in the amount in the bag.

(iv) We find it easier to begin by putting a few pebbles into the bag, and thereafter to take a few out. It is less convenient to begin by taking some out.

(v) Remember that you are dealing with problems such as

$$\begin{aligned} 3 - 5 &= -2 \\ 7 - 3 &= +4 \end{aligned}$$

where the numbers on the left are unsigned numbers and the numbers on the right are signed numbers. The reason for this is that, in

$$3 - 5 = -2,$$

we counted 3 pebbles into the bag (a counting situation, involving unsigned "counting" numbers), then we counted 5 pebbles out of the bag (a counting situation, again), and then we asked about the condition of the bag, in relation to the reference level (a "reference point" situation, involving signed numbers).

(vi) We strongly recommend a careful use of the words *positive*, *negative*, *plus*, and *minus*, and the symbols for them. Small symbols, written high, indicate "positive" and "negative," and are part of the name of the number itself. They do *not* refer to the operations of addition and subtraction. Conversely, large symbols, centered in the line, indicate operations, and are read "plus" or "minus". For example, we would read

$$+3 + -5$$

as

"positive three plus negative five,"

and we would read

$$+7 - -2$$

as

"positive seven minus negative two."

Can you make up a "pebbles-in-the-bag" story for each problem? Can you write the correct signed number to describe what happened in each case?

(11)  $7 - 2 = \underline{\quad}$

- (11) Ellen (or somebody) said "Go!" We put 7 pebbles into the bag. We removed 2 pebbles from the bag. There are now 5 more pebbles in the bag than there were when Ellen said "Go."  
We write

$$7 - 2 = +5,$$

which we read as

"seven minus two equals positive five."

(12)  $5 - 4 = \underline{\quad}$

- (12) Joan (or somebody) said "Go!" We put 5 pebbles into the bag. We took 4 pebbles out of the bag. At this point, we have one pebble more in the bag than we had when Joan said "Go!"  
We write

$$5 - 4 = +1,$$

which we read as

"five minus four equals positive one."

(13)  $3 - 4 = \underline{\quad}$

- (13) Kathy (or somebody) said "Go!" We put 3 pebbles into the bag. We took 4 pebbles out of the bag. There were fewer pebbles in the bag than when Kathy said "Go!" How many fewer? One fewer.

Consequently, we write

$$3 - 4 = -1,$$

which we read as

"three minus four equals negative one."

(Hopefully, by now you and your students are beginning to get some feeling as to how *counting situations* and *reference point situations* differ from one another, and how each of them works out.)

(14)  $2 - 10 = \underline{\quad}$

- (14) Bernice (or somebody) said "Go!" We put 2 pebbles into the bag. We took 10 pebbles out of the bag. We had fewer pebbles in the bag than there were when Bernice said "Go!" How many less? Eight less.

So we write

$$2 - 10 = -8$$

which we read as

"two minus ten equals negative eight."

(15)  $6 - 6 = \underline{\quad}$

- (15) Janet said "Go!" We put 6 pebbles into the bag. We took 6 pebbles out of the bag. Now, at this point, do we have *more* pebbles in the bag than there were when Janet said "Go!" or do we have less, or what? We have the same number.

We write

$$6 - 6 = 0,$$

which we read as

"six minus six equals zero."

You can observe essentially this sequence in the film entitled "A Lesson with Second Graders."

(16)  $3 - 2 = \underline{\quad}$

- (16) Jill said "Go!" We put 3 pebbles into the bag. We took 2 pebbles out of the bag. Now, are there more pebbles in the bag than there were when Jill said "Go!" or are there less, or what? There are more. How many more? One more.

Consequently, we write

$$3 - 2 = +1,$$

which we read as

"three minus two equals positive one."

Notice that this differs from the pure *counting* situation where we would say

$$3 - 2 = 1.$$

In the present situation, does

$$3 - 2 = '1$$

mean that there is now *exactly one pebble in the bag*? No! By no means! Indeed, we do not know how many pebbles there are in the bag.

What *does*

$$3 - 2 = +1$$

mean? It means there is now one more pebble in the bag than there was when Jill said "Go!" It is *this* distinction which causes

$$3 - 2 = 1 \quad \text{and} \quad 3 - 2 = '1$$

to describe two quite different situations.

(17)  $9 - 10 = \underline{\hspace{2cm}}$

(17) Eloise said "Go!" We put in 9. We took out 10. There is now one pebble less in the bag than there was when Eloise said "Go!"

Consequently, we write

$$9 - 10 = '1,$$

which we read as

"nine minus ten equals negative one."

(18)  $2 + 3 - 1 = \underline{\hspace{2cm}}$

(18) Marion said "Go!" We put 2 pebbles into the bag. We put 3 pebbles into the bag. Then we took 1 pebble out of the bag. Now (since altogether we put in 5 and took out 1) there are 4 more pebbles in the bag than there were when Marion said "Go!"

Consequently, we write

$$2 + 3 - 1 = '4,$$

which we read as

"two plus three minus one equals positive four."

(19)  $5 - 4 - 1 = \underline{\hspace{2cm}}$

(19) Jerrold said "Go!" We put 5 pebbles into the bag. Then we took 4 pebbles out of the bag. Then we took 1 pebble out of the bag. Now, at this point, are there more or less pebbles in the bag than there were when Jerrold said "Go!"? There are the same number.

Consequently, we write:

$$5 - 4 - 1 = 0,$$

which we read as:

"five minus four minus one equals zero."

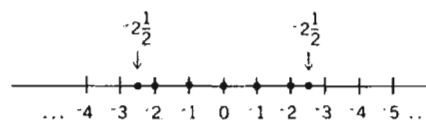
(20)  $4 - 3 - 2 = \underline{\hspace{2cm}}$

(20) Try this one yourself.

(21) Can you mark these numbers on a number line?

$$\{0, -1, -2, -2\frac{1}{2}, -1, -2\frac{1}{2}, -2\}$$

(21)



(22) Can you mark some more numbers on a number line?

(22) This should pose no problems.

(23) Using Descartes' idea of **crossed number lines**, can you mark these points on a graph?

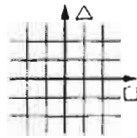
A: (0, 0)

B: (-2, 3)

C: (-1, -1)

D: (-2, -3)

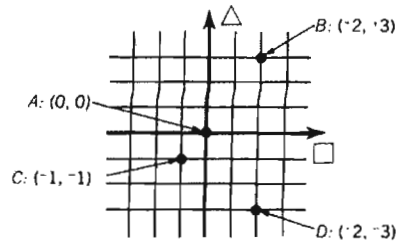
(24) Can you play our game of "four-in-a-row" on this board?



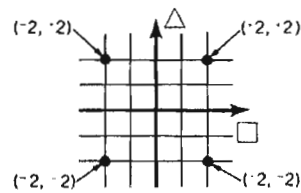
[page 15]

What is the **biggest** number you can put in the □? What is the **smallest**? What is the **biggest** number you can put in the △? What is the **smallest**?

(23)



(24) This version of tic-tac-toe uses signed numbers. With the axes as drawn (and remember, we count from the **heavy lines** or "axes," counting "city blocks" rather than "intersections"), the corners of the board are:



Consequently, **numbers greater than positive two** will result in *illegal moves* (and hence a wasted turn).

To answer what is the "smallest" legal number for this version of tic-tac-toe, we need to agree first on what we mean by "smallest." Which shall we call "smaller,"  $\frac{1}{100}$  or  $-5$ ? Which shall we call "smaller,"  $-\frac{1}{2}$  or  $1,000,000$ ? Both answers make sense, but for the "linear ordering" used by mathematicians, we agree that

$$A < B$$

shall be proclaimed to mean

A lies to the left of B on the number line.

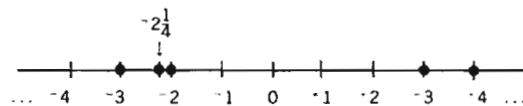
With this interpretation,  $-3 < -2$  and  $-2 < 1$  and  $-1 < 0$ . We can now say that  $-2$  is the smallest legal number for this version of tic-tac-toe. Any number smaller than  $-2$  (for example,  $-3$ ) would result in an illegal move and a wasted turn.

(25) Can you mark these numbers on a number line?

$$\{-3, -2, -3, 2\frac{1}{2}, -4\}$$

- (a) Is '3 more or less than -4?
- (b) Is '3 to the right or left of '4?
- (c) Is -3 more or less than '3?
- (d) Is '3 to the right or left of '3?
- (e) Is -2 more or less than  $-2\frac{1}{2}$ ?
- (f) Is -2 to the right or left of  $-2\frac{1}{2}$ ?

(25)



- (a) '3 is less than -4.
- (b) '3 is to the left of '4.
- (c) -3 is less than '3.
- (d) -3 is to the left of '3.
- (e) -2 is more than  $-2\frac{1}{2}$ .
- (f) -2 is to the right of  $-2\frac{1}{2}$ .

(26) Do you know what  
 $<$   
 means on the number line?

(26) As mentioned in answer to question 24,

$$A < B$$

means

$A$  lies to the left of  $B$  on the number line.

(27) Which statements are true and which are false?

(27)

(a)  $+5 < +7$

(a) True

(b)  $0 < +1$

(b) True

(c)  $0 < \frac{1}{2}$

(c) True

(d)  $0 < \frac{1}{10}$

(d) True

(e)  $+\frac{1}{2} < +\frac{1}{3}$

(e) False

(f)  $\frac{1}{10} < +\frac{1}{3}$

(f) True

(g)  $-1 < 0$

(g) True (That is, negative one is less than zero.)

(h)  $-2 < -1$

(h) True (That is, negative two is less than negative one.)

(i)  $-1 < +1$

(i) True (That is, negative one is less than positive one.)

(j)  $+2 < -2$

(j) False

(k)  $+2 < -3$

(k) False

(l)  $-5 < 0$

(l) True (That is, negative five is less than zero.)

(m)  $-5 < \frac{1}{2}$

(m) True

(n)  $+\frac{1}{3} < 0$

(n) False

(o)  $-5 < -1$

(o) True

(p)  $-1000 < 0$

(p) True

(q)  $-1000 < +1$

(q) True

(r)  $+3 < -1000$

(r) False

(s)  $0 < -1000$

(s) False

(t)  $-1 < -1$

(t) False

(u)  $+7 < +7$

(u) False

(v)  $+7 < -10$

(v) False

(w)  $-10 < -7$

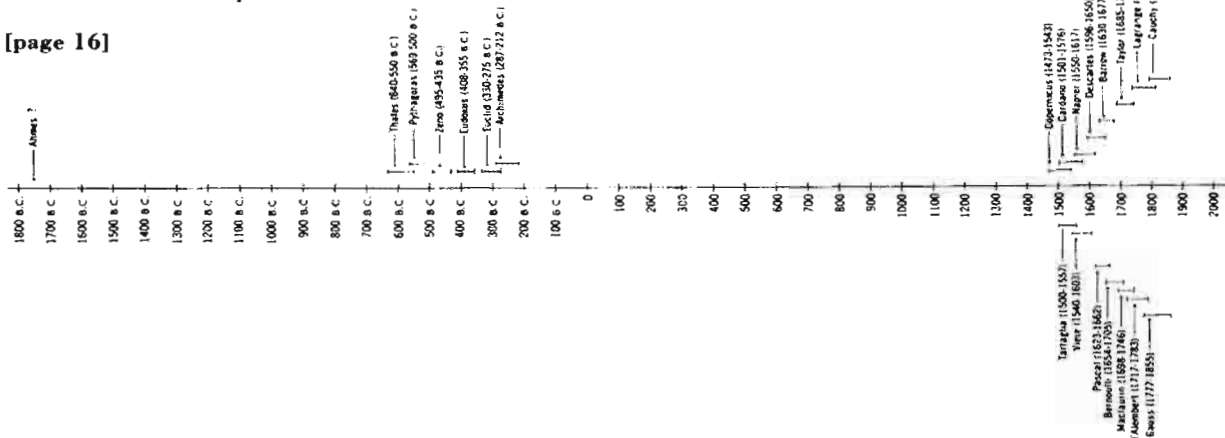
(w) True

(28) Erik says that dates for B.C. and A.D. almost work like the number line, but not quite! What do you think?

(28) Erik is correct. The "date line" unfortunately goes from 1 B.C. to 1 A.D., omitting the year zero which should fall in between. However, by the time you are a century or more away from 1 B.C., this relative error is not of much importance. Would you think much differently of Beethoven if he had been born in 1769 A.D. instead of 1770 A.D.? Similarly, an 83-year-old man is "practically 82" and "practically 84." However, a two-year-old child is neither "practically one" nor "practically three," and a one-year-old child is not "practically a newborn babe." A trip of 101 miles is "practically 100 miles," but a walk of 100 yards is not "practically the same thing as a walk of 5580 feet." The addition of one to a large number doesn't matter very much, but the addition of one to a small number matters a great deal.

(29) Even though the dates are only almost like a number line, Erik says that "a sort of number-line picture" helps him to understand the history of Western civilization. Here is the picture Erik made:

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(29) Erik's chart can be very valuable indeed.

Erik has considered only mathematicians. You can, similarly, make charts using only poets, only playwrights, only scientists, only explorers, or only composers, etc. In nearly every case, the chart will be most suggestive.

If you consider only Western civilization, you will tend to get clusters around -600 to -300, and then from +1500 to the present. Some of this, of course, is an artifact of our method of recording and studying the past; nonetheless, it poses intriguing questions that cry out for explanation.

Using these number-line charts, your students can discover, for themselves, the Renaissance! Much is said in favor of unifying our studies; here is an excellent opportunity to illuminate history by a simple use of mathematics! You may want to consult Appendix A, Eves (151), Kramer (154), and Lloyd (156). Your students might also enjoy reading Eileen Power's *Medieval People* (Doubleday-Anchor).

(30) Don says Erik only considered mathematicians, and he left out a great many mathematicians, even at that. Can you add some of the following mathematicians, to Erik's chart?

J. W. Alexander	(1856-1915)
P. Alexandroff	(1896- )
Emil Artin	(1898- )
Stefan Bergman	(1898- )
Friedrich Wilhelm Bessel	(1784-1846)
R. H. Bing	(1902- )
George David Birkhoff	(1884-1944)
George Boole	(1815-1864)
George Cantor	(1845-1918)
Constantin Carathéodory	(1873-1950)
Arthur Cayley	(1821-1895)
W. K. Clifford	(1845-1879)
J. W. R. Dedekind	(1831-1916)
Albert Einstein	(1879-1955)
Gottlieb Frege	(1848-1925)

(30) Compare answer to question 29.

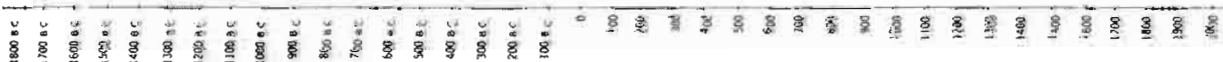
Obviously, if you don't enjoy the idea of combining a little mathematics and a little history, you can omit these "historical" questions.

Évariste Galois	(1811-1832)
Kurt Gödel	(1906- )
Herman Grassmann	(1809-1877)
John Graunt	(1620-1674)
Edmond Halley	(1656-1742)
Sir William Rowan Hamilton	(1805-1865)
Felix Hausdorff	(1868-1942)
J. B. van Hemholtz	(1821-1894)
David Hilbert	(1862-1943)
Bela van Kérekjártó	(1898-1946)
	[page 17]
Felix Klein	(1849-1925)
Andrei Kolmogorov	(1903- )
Henri Leon Lebesgue	(1875-1941)
Solomon Lefschetz	(1884- )
Deane Montgomery	(1909- )
R. L. Moore	(1882- )
John von Neumann	(1903-1957)
Amalie Emmy Noether	(1882-1935)
G. Peano	(1858-1932)
Jules Poincaré	(1854-1912)
George Polya	(1887- )
L. S. Pontriagin	(1908- )
Srinivasa Ramanujan	(1887-1920)
Georg Friedrich Riemann	(1826-1866)
J. B. Rosser	(1907- )
Bertrand Arthur Russell	(1872- )
Waclaw Sierpinski	(1882- )
T. A. Skolem	(1887- )
M. H. Stone	(1903- )
Alfred Tarski	(1902- )
Oswald Veblen	(1880- )
Karl Theodor Weierstrass	(1815-1897)
Andre Weil	(1906- )
Hermann Weyl	(1885-1955)
Alfred North Whitehead	(1861-1947)
Norbert Wiener	(1894-1964)
R. L. Wilder	(1896- )
E. F. F. Zermelo	(1871- )

(31) Ellen says Erik's chart is not big enough to get all the names in. Can you make a chart that is big enough? (Use the same scale from 1800 B.C. through 2000 A.D.)

(31) through (32) Compare answer to question 29.

(32) Bill said it would be easier just to mark a dot to show when each mathematician was born. Bill began his chart with dots for Ahmes, Thales, Pythagoras, Polya, Dedekind, Graunt, Peano, and Veblen. At this stage, Bill's chart looked like this:



Can you mark on Bill's chart all the other mathematicians mentioned in this chapter?

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(33) Harold said these charts seem to say something about history. He made a new chart, marking the dates of birth of the following musicians. What did Harold's chart look like?

Johann Sebastian Bach	(1685-1750)
Karl Philipp Emanuel Bach	(1714-1788)
Béla Bartók	(1881-1945)
Ludwig von Beethoven	(1770-1827)
Leonard Bernstein	(1918- )
Johannes Brahms	(1833-1897)
Elliott Cook Carter	(1908- )
François Frédéric Chopin	(1810-1849)
Aaron Copeland	(1900- )
Claude Debussy	(1862-1918)
Giovanni Gabrielli	(1554-1612)
George Gershwin	(1898-1937)
George Friedrich Handel	(1685-1759)
Roy Harris	(1898- )
Franz Joseph Haydn	(1732-1809)
Wolfgang Amadeus Mozart	(1756-1791)
Sergei Sergeevich Prokofiev	(1891-1953)
Sergei Wassilievitch Rachmaninoff	(1873-1943)
Alessandro Scarlatti	(1659-1725)
Domenico Scarlatti	(1683-1757)
Franz Peter Schubert	(1797-1828)
Robert Schumann	(1810-1856)
Dmitri Dmitrievich Shostakovich	(1906- )
Igor Stravinsky	(1882- )
Petr Ilich Tchaikovsky	(1840-1893)
Antonio Vivaldi	(1675-1741)
Richard Wagner	(1813-1883)
William Walton	(1902- )
Kurt Weill	(1900-1950)

(34) Ellen made a chart showing birth dates of painters, sculptors, and architects. What did Ellen's chart look like?

(35) Nancy made a chart showing birth dates of playwrights, writers, and poets. What did Nancy's chart look like?

(36) Andy made a chart showing birth dates of explorers. What did Andy's chart look like?

(37) Dick made a chart showing birth dates of scientists. What did Dick's chart look like?

(38) Do these charts suggest anything? How do you explain it?

(39) Do you know the date when cities first began to appear? What sort of chart can you make from the beginning dates of various cities?

(33) Compare answer to question 29.

(34) through (37) If you wish, you can use this as a research assignment for your students. You might also ask students to make a chart of some inventions or discoveries. References McClelland (50) and Woodward (161) will also be useful for such an assignment.

(38) Compare answer to question 29.

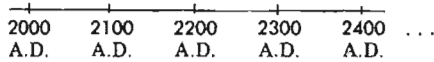
(39) We leave this up to you and your students.



(40) Do you know what people mean when they speak of the Renaissance?

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(41) What do you think these charts will look like for the following part of the number line?



(42) You may be interested in studying the changes in the population of Europe. A chart can help here, too.

(43) Here is a table for the population of London. You may want to make a chart of this.

Date	Population
1801	1,088,000
1811	1,259,000
1821	1,504,000
1831	1,778,000
1841	2,073,000
1851	2,491,000
1861	2,291,000
1871	3,336,000
1881	3,881,000
1891	4,266,000
1901	4,563,000
1911	4,541,000
1921	4,498,000
1931	4,408,000
1951	3,353,000

Population of London

(44) Do you have any theories on why the graphs look like this? You might try to find the dates of the following events to check against your theories.

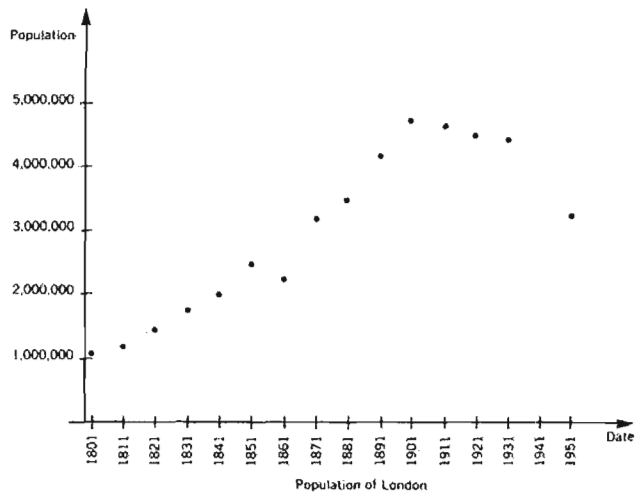
- Founding of Oxford University
- Founding of Cambridge University
- Founding of University of Paris
- Discovery of the source of the Nile
- French Revolution
- California gold rush
- Discovery of America
- First European settlement in South America
- First trip around the world
- Marco Polo's birth
- First universal compulsory education in the United States
- Founding of Hopkins Grammar School
- First railroad across United States

(40) I hope the number-line pictures of the preceding few questions make this all too clear.

(41) Your guess is as good as anyone else's. Assuming a continuation of recent trends, you can fill in a rather spectacular-looking picture.

(42) Another possible research assignment.

(43) We can make the "Population of London" graph as follows:



One difficulty in this chapter is the task of coping with numbers of the size we need for the study of history. (In some cases you may want to use logarithmic graph paper.) For a particularly fine example of an attempt to cope with very large numbers, consult Appendix A, Ardrey (147), pages 208-213.

(44) This is a very open-ended, library research question. Perhaps one or more of your students (or, better, a committee of them) may wish to prepare a special report on this. Several helpful references are listed in the *Student Discussion Guide*. For an interesting discussion of what is (at the time of this writing) the earliest known record of prime numbers, refer to: Jean de Heinzelin, "Ishango," *Scientific American* (June, 1962, pp. 105-116). As to the theories your students may suggest, we make no predictions.

For a general view of theory construction, it may be worth mentioning the important work done by J. Richard Suchman, of the University of Illinois, in connection with his Inquiry Training program. Professor Suchman emphasizes that—ostensibly, at least—science does not recognize any "official" theories. Any theory is admissible to the extent that it explains the known facts and is capable of meeting the demands made upon it.

Crusades  
First European contacts with China  
Stradivarius' birth  
United States Revolutionary War  
United States Civil War  
World War I  
World War II  
Date of discovery of prime numbers, etc.  
(Belgian Congo)  
Date of *Australopithecus Africanus*

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#### Bibliography

Ardrey, R., *African Genesis* (Dell Publishing Company, New York, 1953).

Aylmer, G. E., *A Short History of 17th Century England: 1603-1689* (Mentor Books, New American Library, New York, 1963).

Bell, E. T., *Men of Mathematics* (Simon & Schuster, New York, 1961).

Derry, J. W., *A Short History of 19th Century England: 1793-1868* (Mentor Books, New American Library, New York, 1963).

Eves, H., *An Introduction to the History of Mathematics* (Holt, Rinehart and Winston, New York, 1964).

Harris, R. W., *A Short History of 18th Century England: 1689-1793* (Mentor Books, New American Library, New York, 1963).

Haskins, C. H., *The Rise of Universities* (Cornell University Press, Ithaca, N. Y., 1957).

Leasor, J., *The Plague and the Fire* (Avon Books, New York, 1961).

Moorehead, A., *The White Nile* (Harper and Row, New York, 1960).

Newman, J. R., *The World of Mathematics* (Simon & Schuster, New York, 1962).

Woodward, G. W. O., *A Short History of 16th Century England: 1485-1603* (Mentor Books, New American Library, New York, 1963).

## POSTMAN STORIES

In the preceding chapter we used the "pebbles-in-the-bag" model to introduce signed numbers. That was, presumably, the first time the students encountered the idea of signed numbers.

It is important to remember that the pebbles-in-the-bag model serves (very well) to introduce *signed numbers*. It does *not* introduce the *arithmetic* of signed numbers. Using the pebbles-in-the-bag idea, we can think meaningfully about +2, -3, and so on, but we *cannot add, multiply, or subtract* these new numbers.

What we *can* do with the pebbles-in-the-bag model is to add and subtract *unsigned* ("counting") numbers, and to *express the answer* as a signed number.

Using the pebbles-in-the-bag model, we *can* do these:

$$5 - 3 = +2$$

$$7 - 11 = -4$$

$$3 + 2 - 1 + 7 - 10 = +1$$

$$5 - 8 + 4 - 6 = -5$$

We *cannot* do these:

$$+5 - +3 = +2$$

$$+5 - -3 = +8$$

$$+2 \times +3 = +6$$

$$+2 \times -3 = -6$$

$$-7 + -2 = -9$$

Now, in this chapter, we introduce "postman stories." By the time that we have finished with postman stories, we shall be able to handle the entire arithmetic of signed numbers.

One or two remarks about this chapter may be helpful:

The postman and the housewife behave as in the fantasy novels of Franz Kafka. We have never found this troublesome with children; after all, children enjoy "Superman" and similar fantasies. As long as the teacher is not disturbed by fantasy, the children will not be. Indeed, properly (and *lightly*) handled, fantasy *strengthens* one's hold on reality, rather than weakening it, for we all learn best by contrasts and comparisons.

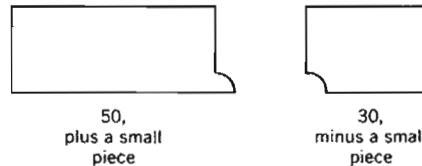
What do the postman stories do? They provide a suitable set of mental symbols which can be "manipulated" mentally so as *always* to suggest the correct answer to problems in the arithmetic of signed numbers. Such mental symbols, described especially in the work of Tolman, Piaget, Aldous Huxley, and Kurt Lewin, deserve more attention than they usually receive [see Appendix A: Flavel (114), Tolman (69), Davis (28), Lewin (129), Huxley (43), and Hoyle (124).] Let me give three examples:

(i) Is it easier to take off your shoes before taking off your socks, or is it easier to take off your socks before taking off your shoes? *You do not need to experiment in the physical world in order to find out.* Why not? Because you have a set of mental symbols which you can "experiment with" inside your head, as it were.

(ii) My poodle, when tied to a tree, runs around the tree until he has no free rope left. Then he doesn't know how to unwind himself, so he howls until someone comes to help (admittedly not an ineffective strategy). You and I, being human beings, have in our minds mental symbols for "dog," "rope," and "tree," with which we can perform a "thought experiment." These mental symbols have a complete cognitive-level set of "rules of dynamics" that makes such "thought experiments" possible. If winding counter-clockwise has shortened the rope, then "unwinding" in the opposite direction will lengthen the free rope.

You and I don't even need to try this out; we *know* it will work. The poodle, evidently, has no such set of mental symbols available to him, so he stands, tied to a very short rope, and howls.

(iii) How much is  $53 + 27$ ? We don't need any algorithm at all to answer; we can use symbols, such as these:



Each small piece is the same size, namely, 3. Therefore, put together, we clearly have

$$50 + 30 = 80.$$

More precisely, here is what we want postman stories to do for us:

Whenever we have a mathematical problem, such as

$$+37 - ^{-}2 = ?,$$

we want the postman stories to provide a corresponding story that will show us what the answer should be.

When we start with a postman story, it is *not* necessary that there be a corresponding mathematical problem, since we mean to use postman stories to explain mathematics, and not conversely.

We shall *begin* by starting with postman stories and then finding corresponding mathematical expressions. (Purists among the audience may object that some of the mathematical expressions are not of normal occurrence, because they confuse symbols for binary and unary operations; but this is unimportant, since when we come to use the stories in actual practice, we shall *always* be *starting with the mathematics* and *seeking an appropriate story*, and never conversely.)

Here is the way we shall work:

When we say "bills," we mean what the gas company, the electric company, and the furniture company send to us. (We do not mean those lovely pieces of paper printed by the folks in Washington and called "ten dollar bills.")

When we say "checks," we mean those lovely things our employer gives us, and our broker sends us, and so on. (We do *not* mean those things you get in restaurants that make you poorer instead of richer.)

Thus, when we *receive a check*, we get richer; when we *give back a check*, we get poorer; when we *receive a bill*, we get poorer; when we *give back a bill*, we get richer.

At this point you may want to read carefully the explanation in the *Student Discussion Guide*. Notice that the "fantasy" behavior has been devised so that the postman stories work out *exactly* as described above, with regard to "receiving" or "giving back" bills and checks. *The stories may sound foolish, but they are precisely and reliably consistent in their logic.* They embody neither contradictions nor "double-counting."

For the postman story	we write
postman <i>brings a check</i> for \$5.00	+ 5
postman <i>takes away a check</i> for \$5.00	- 5
postman <i>brings a bill</i> for \$5.00	+ 5
postman <i>takes away a bill</i> for \$5.00	- 5

Notice that bills are represented by *negative numbers*, checks are represented by *positive numbers*, *bringing* is represented by a "plus" sign, *taking away* is represented by a "minus" sign.

At this point, you may want to view the Madison Project film entitled "Postman Stories." Before you do, it will be well to discuss what you can see in this film.

In making nearly all Madison Project films, we try to show a *new learning experience* of the children—they are confronted with a task they have never met before, and the viewer can watch how the children work their way through this new problem, usually with relatively little help from the teacher. To make such films successful, the children must have adequate previous background (or "readiness") so that it is reasonable to expect that they will succeed in attacking this new problem, but they *must not* have so much "readiness" that the "new" problem isn't really new.

Now, achieving this is not easy. If, on Thursday, the teacher *felt* that the students would be ready for the new task on Friday—and if we could rent TV facilities for videotaping on a few hours notice—the problem of arranging such films would not be too difficult. However, it takes several weeks to arrange TV videotaping facilities.

Consequently, a Madison Project filming session is planned like a "moon shot" from Cape Kennedy: you don't aim *at* the moon; rather, you try to arrange for your space capsule and the moon to arrive at the same future point at the same future time.

We must estimate, well in advance, *when* the students will be ready for the new topic, and hope that the day they're ready for the new topic turns out to be the day the TV cameras are there. Obviously, we sometimes miss.

The film "Postman Stories" is an interesting case. We used a class of so-called "culturally deprived" children, provided by Mr. Ogie Wilkerson and Mr. Cozy Marks, of the St. Louis Public Schools. We *planned* to show how these children learned to match up "postman stories" with corresponding mathematical situations.

Once the cameras started rolling, it became evident that the class had too much readiness for this task—there was too little "new" learning taking place. Consequently, the teacher had to

jump immediately to a "harder" task, where really *new* learning could occur. He turned to the task of graphing

$$(\square \times \square) + (\triangle \times \triangle) = 25,$$

which was entirely new for the class, and which *makes use of* "postman stories."

The result was one of our most successful films. At the beginning, the children give wrong answers to nearly every problem in the arithmetic of signed numbers (saying, for example, that " $-1 \times -1 = 0$ ," and that " $-1 \times -1 = -2$ "). Next, the children use "postman stories" to decide—by themselves!—what the correct answers should be. Toward the end of the lesson, they have gained enough insight into how the arithmetic of signed numbers works so that they give correct answers without recourse to "postman stories"!

Now, *this is just what we want "postman stories" to do!* We want them to provide the children with an "autonomous decision procedure" whereby the child can decide *for himself* what answer he should give in a problem involving the arithmetic of signed numbers.

This film proves—better than anything we could have planned—that *postman stories are capable of providing a foundation for the arithmetic of signed numbers*—for "culturally deprived" children as well as for "culturally privileged" children.

And, notice, nobody told the children any "rules" for working with signed numbers.\*



## CHAPTER 5 Postman Stories

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Jerry wrote a story about a very peculiar postman, who behaved like this:

- (a) He read all of the mail.
- (b) He did not necessarily deliver the mail to the right people. He gave it to anyone he wanted to give it to. (But he remembered who should have received it!)
- (c) Later on he would come back and pick up mail he had misdelivered, apologize, and give it to the right person.

Jerry's story also includes a housewife, who also behaves peculiarly:

- (a) She tries to keep up-to-date in her estimate of how much available money she has.
- (b) She never reads the addresses on the mail she receives (she figures it doesn't do any good anyhow, because the postman delivers them to whomever he wants), and she never reads the name on bills and checks (but she reads the amount and keeps her records up-to-date!).

## ANSWERS AND COMMENTS

\*Cf. Max Beberman, *An Emerging Program of Secondary School Mathematics*, p. 25 (Harvard University Press, Cambridge, Mass., 1958):

Teachers of the conventional course in beginning algebra recognize the fact that students are very quick in discovering a rule for adding directed numbers [i.e., "signed numbers"]. In fact, the usual rule stated in textbooks is a necessarily complicated description of an algorithm . . . Any student capable of learning algebra in the first place will have invented this algorithm. Any student who is able to interpret the textbook description is also able to carry out the algorithm for adding without using the text description. [I have added the italics—R.B.D.] Hence, our earliest opportunity for an important discovery in the UICSM program occurs in connection with the rule for adding directed numbers. All students succeed in this first attempt. [Italics again added—R.B.D.]

Telling students, "rules" for the arithmetic of signed numbers is an exercise in utter futility. Adults of our acquaintance who were told such rules in school nearly always repeat them—and use them—*incorrectly* at this point in their adult life. When we show these same adults the "postman story" model, they become able to get correct answers without recourse to (incorrectly) memorized rules.

Jerry's story involves bills, like

3, '1, '5, '100, '10,

and checks, like

'2, '7, '5, '100, '9.

(1) Do you know what Jerry means by a check? Who might send you a check?

(1) This question is intended to emphasize that when we receive a check, we become richer.

(2) Do you know what Jerry means by a bill? Do you like to get bills? Who might send you a bill?

(2) This question is intended to clarify our present use of the word bill: when we receive a bill, we become poorer.

Jerry's postman sometimes brings checks

+ '3,

and he sometimes comes and takes away a check (that was really for someone else)

- '10.

The postman sometimes brings bills

+ '7,

and he sometimes takes away a bill (that was really for somebody else).

(3) Does it make you happy or sad when the postman brings a bill?

(3) Sad

(4) Does it make you happy or sad when the postman takes away a bill?

(4) Happy

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(5) Does it make you happy or sad when the postman brings a check?

(5) Happy

(6) Does it make you happy or sad when the postman takes away a check?

(6) Sad

(7) Jerry said, "On Monday morning, the postman brought the housewife a check for \$3 and a check for \$5."

(7) The housewife thought she was richer, by \$8. Consequently, she changed her estimate of her available funds upward \$8; if, say, she had thought she had \$120 available to her, she now changed this to \$128.

+ '3 + '5

As a result of the postman's visit on Monday morning, did the housewife think she was richer or poorer? How much richer or how much poorer?

~~\$120~~  
\$128

(8) Can you write a single signed number showing how much richer or poorer the housewife thought she was?

(8) We could write + '3 + '5 = '8.

+ '3 + '5 =

(9) Geoffrey's father says that mathematicians sometimes leave off the first "+" sign and write merely

(9) '3 + '5 = '8. This is the form which occurs normally in mathematics.

'3 + '5.

Can you write a single signed number that names the same amount as '3 + '5?

'3 + '5 =

(10) The housewife thought she had \$120 uncommitted and available before the postman came Monday morning. How did she change her records as a result of the postman's visit Monday morning?

\$120  
 ← ?

(11) Gloria says the housewife's records should look like this:

\$120  
 \$130

Do you agree?

Can you make up a postman story for each problem? What answer do you get?

(12)  $-2 + '7$

(10) ~~\$120~~  
 \$128

(11) **No. Compare answer to question 10.**

(12) **On, say, Tuesday morning, the Postman came and brought**

a check	↓ '2 + '7
for \$2	↓ '2 + '7
and he also brought	↓ -2 + '7
a check	↓ '2 + '7
for \$7.	↓ '2 + '7

As a result of his visit on Tuesday morning, the housewife believes herself to be *richer* by \$9. She will revise her estimated available funds upward by \$9. We could write

$$'2 + '7 = '9.$$

(13)  $'2 + -1$

(13) **On, say, Thursday morning, the Postman brought**

a check	↓ '2 + -1
for \$2	↓ '2 + -1
and he also brought	↓ '2 + -1
a bill	↓ '2 + -1
for \$1.	↓ '2 + -1

As a result of the postman's visit on Thursday morning, the housewife believes herself to be *richer* by \$1. She will revise her estimate of available funds upward by \$1. We can write

$$'2 + -1 = '1.$$



(14)  $-5 + -2$

(14) The postman brought

a bill	↓	$-5 + -2$
for \$5	↓	$-5 + -2$
and he also brought	↓	$-5 + -2$
a bill	↓	$-5 + -2$
for \$2.	↓	$-5 + -2$

As a result of the postman's visit on Friday morning, the housewife believed herself poorer by \$7. She *decreased* her estimate of available funds by \$7. We could write

$$-5 + -2 = -7.$$

(15)  $+3 + -4$

(15) On, say, Saturday morning, the postman brought a check for \$3 and he also brought a bill for \$4.

As a result of the postman's visit on Saturday morning, the housewife believes herself to be poorer by \$1. She revises her estimate of available funds *downward* by \$1. We can write

$$+3 + -4 = -1.$$

The reason for the emphasis on the *day* when the postman visits will become clear in answer to question 16 below.

(16)  $+9 - -2$

(16) On, say, Monday morning, the postman came, and brought a check for \$9 and took away a check for \$2. The postman remarked, "I sure hope you weren't planning on spending that check for two dollars. It's really for Mrs. Wilson. If you'll give it back to me, I'll run over and deliver it to her right now."

As a result of the postman's visit on this Monday morning, Mrs. Housewife believes herself to be *richer* by \$7. She revises her estimate of available funds upward by \$7, say,

~~\$150~~  
\$157

We could write

$$+9 - -2 = +7.$$

Why have we put so much stress on the *time* factor of the postman's visits? This problem shows the reason. Students will sometimes confuse the problem

$$+2 + (+9 - -2) = +9$$

with the present problem

$$+9 - -2 = +7.$$

The student may ask, "Why isn't Mrs. Housewife richer by nine dollars? She just *got* a check for two dollars, then gave it back. Why should that have any effect?"

The answer, of course, is that Mrs. Housewife received the check for \$2 sometime last week, and has *already included it in her estimate of available funds*. Consequently, when she has to *return* the check for \$2, she must reduce her estimate by the corresponding \$2.

The use of the *time* factor lets us distinguish easily between

$$+2 + (+9 - +2) = +9$$

and

$$+9 - +2 = +7.$$

Writing  $+9 - +2 = +7$  describes what happened as a result of the postman's visit on this Monday morning.

By contrast,  $+2 + (+9 - +2) = +9$  describes a *combination of part of last week's transactions*

$$+2 + (+9 - +2) = +9$$

↑

together with the result of the postman's visit *this morning*:

$$+2 + \underbrace{(+9 - +2)}_{\uparrow} = +9.$$

With a little practice, plus careful attention to details, I believe you will find this works both easily and reliably. The use of *time* facilitates distinctions such as those above.

$$(17) \quad - +5 - +2$$

(17) On, say, Tuesday morning, the postman came and took away a check for \$5 and he also took away a check for \$2.

(The postman said, "I hope you haven't been making plans for spending those checks. They really belong to Mrs. Cohen. If you'll give them back to me, I'll run over and give them to Mrs. Cohen as soon as I'm through with work.")

Unfortunately, Mrs. Housewife had, as usual, counted these checks into her estimate of available funds. Consequently, as a result of the postman's visit this morning, she had to *decrease* her estimate of available funds by \$7. Whereas she had thought she had \$157, she changed this now to \$150.

$$\begin{array}{r} \$150 \\ \$157 \\ \$150 \end{array}$$

We could write

$$- +5 - +2 = -7.$$

This, also, is a notation that will not occur often in our mathematical work; *strictly speaking*, it also confuses binary and unary operations. However, even though, in this sense, we *need* not explain such "foolish" notations, we are in fact *able* to do so if we choose.

Similarly, the notation shown in answer to question 18 is a notation that we shall not ordinarily encounter in our mathematical work; nonetheless, we *can* explain it if we wish.

$$(18) \quad - +5$$

(18) The postman came Wednesday morning. He took away a bill for \$5. Now, the housewife had already provided for this bill in her estimate of available funds. Consequently, when the postman came this morning and told her this bill was not really for her,

she breathed a sigh of relief and *increased* her estimate of available funds by \$5.

~~\$150~~  
~~\$157~~  
~~\$150~~  
 \$155

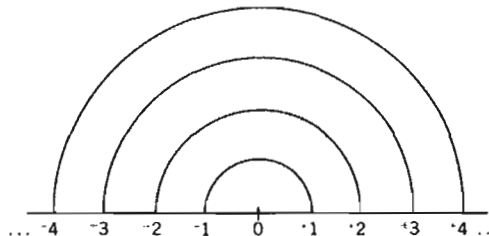
We could write

$$- 5 = +5.$$

Actually, when we come to the notation for *opposites* or *additive inverses* (see *Discovery*, Chapter 40, or *Explorations*, Chapters 11 and 15) we shall write this as

$$^{\circ}(-5) = +5,$$

and represent the unary operation of "finding the additive inverse" by a "rainbow" picture:



To find the "opposite" or "additive inverse" of a number, you "go to the opposite end of the rainbow." You need not worry about this matter at this point; we shall return to it later, in Chapter 11.

(19)  $- -1 - -5$

(19) On Thursday morning, the postman came and

took away	↓	- 1 - 5
a bill	↓	- 1 - 5
for \$1	↓	- 1 - 5
and he also took away	↓	- 1 - 5
a bill	↓	- 1 - 5
for \$5.	↓	- 1 - 5

The housewife had, of course, already provided for both of these bills in her estimate of available funds. When she found out that those bills were not really for her, she revised her estimate of available funds *upward* \$6.

~~\$155~~  
 \$161

For the original problem  $-1 - 5$  we could write

$$-1 - 5 = -6.$$

(Roughly translated, this says that taking away a bill for \$1 and a bill for \$5 makes you richer by \$6.)\*

(20)  $+10 - 100$

(20) The postman arrived Friday morning. He brought

a check	↓ +10 - 100
for \$10	↓ +10 - 100
and he took away	↓ +10 - 100
a bill	↓ +10 - 100
for \$100.	↓ +10 - 100

The check which he brought was, of course, a new matter; but the housewife had *already allowed* for the \$100 bill in her estimate of available funds. When she found that this bill wasn't really for her anyhow, she breathed a large sigh of relief and, combining the morning's two transactions, revised her estimate of available funds upward by \$110.

~~\$161~~  
\$271

We could write

$$+10 - 100 = +110.$$

\*Miss Katie Reynolds, a teacher of the fifth and sixth grades in the At-tucks School in St. Louis (which is part of Dr. Samuel Shepard's well-known "Banneker District"), has developed the most effective method for teaching Postman Stories that any of us on the Project has ever seen. Her method works so smoothly that she is able to teach this topic easily and successfully to an entire class of culturally deprived children whose school performance might ordinarily be quite marginal. Miss Reynolds' device is to use index cards to represent checks, and to use a piece of paper (with an appropriate notation written on it) to represent a bill, so as to gain the advantage of clear visual imagery in relation to Postman Stories. But her particularly ingenious idea is to introduce a "Bill Bag." Whenever the housewife receives a bill for, say, \$7, in order to be sure that she will have the money available to pay it, she does the following: she takes seven index cards representing \$1 each—or some other combination of index cards representing checks that total \$7—wraps the bill around them, puts a rubberband around this, and puts this into her "Bill Bag." The great advantage of Miss Reynolds' method appears when the postman comes to take back a bill, for when he tells the housewife that that \$7 bill was not for her, she reaches into her Bill Bag, takes out this little package with the elastic around it, undoes the package, gives the bill back to the postman, and is now quite visably richer by \$7. Each child can see for himself "where the \$7 comes from."

Some children seem to require direct visual experience as a foundation for building abstract concepts. The "Bill Bag" method which Miss Reynolds developed, in cooperation with the principal of her school, takes much of the abstract terror out of problems like

$$+2 - 7 = -9$$

by giving the child a very clear visual experience. The +2 is *there*, represented by index cards the postman has just handed her. The "extra seven dollars" is *there*, just freed from captivity in the "Bill Bag"; putting them together, the child sees that the housewife is "richer by nine dollars."

(21) - '100

(21) Postman took away a check for \$100. Housewife is "poorer" by \$100:

$$- '100 = '100.$$

(22) '10 - '100

(22) Postman brought a check for \$10, took away a check for \$100. As a result of his visit this morning, the housewife is "poorer" by \$90. We could write

$$'10 - '100 = '90.$$

(23) - '7

(23) Took away bill for \$7. Since the housewife had made provision for this bill in computing her estimate of available funds, when she found out the bill was not for her she revised her estimate upward by \$7:

$$- '7 = '7.$$

(24) '5 - '2

(24) Postman brought a check for \$5 and took away a bill for \$2.

$$'5 - '2 = '7.$$

## POSTMAN STORIES FOR PRODUCTS

Experienced teachers will not doubt that "the product of two negative numbers" is one of the most "mysterious" items in the traditional curriculum. The traditional curriculum approached the arithmetic of signed numbers by stating rules which students were asked to memorize.

Those memorized rules did nothing to dispel the mystery of it all, and with the passing of time most people forgot the precise form of the rules, replacing them with incorrectly recalled substitutes that led to wrong answers.

The "rules" approach *did not work*. (Ask adults of your acquaintance to perform some problems in the arithmetic of signed numbers if you want to see what we mean.) Modern curricula have tended to replace these fallible "rules" with something quite different: namely, *mental imagery* that suggests the correct answer.

There is a growing literature dealing with such imagery, and much more will probably appear in the near future. See Appendix A: Davis (28), (170); Tolman (69); Beberman (103); Sanders (64); Flavell (114); and Brown (105).

Now, teachers may, at first, feel uncomfortable with such imagery. They may ask if "truth" in mathematics should not be made to depend upon *formal mathematical logic*, and not upon "mental imagery."

Actually, mathematical "truth" appears in two forms:

(i) "Heuristic truth," where we *believe* a statement to be true for nonlogical reasons.

(ii) "Logical truth," where we support a statement by a careful logical proof.

To neglect either of these aspects is unwise. "Heuristic truth" guides our *intuition*; it enables us to *guess* which things may be worth trying to prove, and it reassures us that the whole affair "makes sense."

"Logical truth" is often too complicated to guide our intuition; "logic" functions almost like a giant computer, that tells us which statements must be labeled T and which must be labeled F, but it does not tell us *why*, except in the most formalistic and remote sense.\*

It seems very likely that heuristic thought more often than not depends upon "models" or appropriate mental imagery.

Actually, we might distinguish four *kinds* of reasons for believing a mathematical statement to be "true":

(i) A meaningful model suggests the truth of the statement.

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\*Compare Jerome Bruner, *The Process of Education*, pp. 13-14:

"The third theme [of this book] involves the nature of intuition—the intellectual technique of arriving at plausible but tentative formulations without going through the analytic steps by which such formulations would be found to be valid conclusions. Intuitive thinking, the training of hunches, is a much-neglected and essential feature of productive thinking not only in formal academic disciplines but also in everyday life. The shrewd guess, the fertile hypothesis, the courageous leap to a tentative conclusion—these are the most valuable coin of the thinker at work, whatever his line of work. Can school children be led to master this gift?"

(ii) The statement is a simple *agreement*—as when we agree that the first-named coordinate shall refer to the horizontal axis and the second-named to the vertical axis (the rule for substituting is also in this "agreement" category).

(iii) A consistent *pattern* seems to suggest the truth of the statement.

(iv) A logical proof of the statement exists.\*

By the time a student has completed grade 9, we would hope that he could give both intuitive *and* logical reasons for the statement

$$^{-}1 \times ^{-}1 = ^{+}1.$$

[The logical proof depends primarily upon the *distributive law*

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla).]$$

However, considerable experience with children has convinced us that, below grade 9, the "model" approach—specifically, postman stories—works out far more satisfactorily than any of the other approaches. Even for ninth-graders, the postman stories—once they have been mastered—are often the most effective "explanation" of the behavior of signed numbers.

In this chapter we shall take our postman model, which we have already used for addition and subtraction, and extend it to deal with multiplication. Our stories for multiplication will necessarily be different from our stories for adding and subtracting, because the role of *units* (or "dimensions") is different.

In adding, you (ordinarily) want similar units for each term. That is, if

$$\begin{array}{c} ^{-}3 + ^{-}2 \\ \uparrow \\ \text{this is "dollars"} \end{array}$$

then

$$\begin{array}{c} ^{-}3 + ^{-}2 \\ \uparrow \\ \text{this should also be "dollars."} \end{array}$$

However, in multiplying, if both units were "dollars,"

$$\begin{array}{cc} ^{+}2 \times ^{+}3 \\ \uparrow \quad \uparrow \\ \text{"dollars"} \quad \text{"dollars"} \end{array}$$

the "answer" would be in "dollars squared" (as in the case, for example, of "inches" and "square inches"). This would be nonsense. Consequently, we *cannot* interpret *both* factors as dollars.

We shall, instead, interpret products in such a way that the *second* factor will be interpreted as money—i.e., as a bill or a check.

$$\begin{array}{c} ^{+}2 \times ^{+}3 \\ \uparrow \end{array}$$

\*Actually, on a deeper level, "logic" is merely one more instance of "pattern." We observe, for example, the general appropriateness of some logical rule (for example, *modus ponens*), and we thereafter proclaim this rule—that is to say, we make consistent use of this naturally occurring pattern.

We can use the *first* factor to tell us *how many times* the postman brings the item or takes it away.

$$\begin{array}{c} +2 \times +3 \\ \uparrow \end{array}$$

Specifically, these following examples show the four possibilities with multiplication.

(i)  $+2 \times +3$

The postman brings	↓ +2 × +3
two	↓ +2 × +3
checks	↓ +2 × +3
for \$3 each.	↓ +2 × +3

As a result of this visit, you are "richer" by \$6. Hence we write

$$+2 \times +3 = +6.$$

(ii)  $+2 \times -5$

The postman brings	↓ +2 × -5
two	↓ +2 × -5
bills	↓ +2 × -5
for \$5 each.	↓ +2 × -5

As a result of this visit, the housewife revises her estimate of un-committed available funds

downward	↓ +2 × -5 = -
by \$10.	+2 × -5 = -10
	↑

We can write

$$+2 \times -5 = -10$$

(read: positive two times negative five equals negative ten).

(iii)  $-3 \times +4$

takes away	↓ -3 × +4
three	↓ -3 × +4
checks	↓ -3 × +4
for \$4 each.	↓ -3 × +4



He says: "I hope you weren't planning to spend those three checks I brought last week. They're not for you—they're for Miss Parsons. If you'll give them back to me, I'll run over and give them to Miss Parsons as soon as I get through work this afternoon."

As a result of the Postman's disappointing visit this morning, Mrs. Housewife must revise downward her estimate of uncommitted available funds:

$$^{-}3 \times ^{+}4 = ^{-}12.$$

(iv) When you *receive bills*, that's sad. When you *give back* bills, that's good! For the case

$$^{-}2 \times ^{-}6$$

our story goes like this:

The postman came on a warm and sunny April morning—Thursday morning, as a matter of unimpeachable fact—and

took back	↓ ^{-}2 \times ^{-}6
two	↓ ^{-}2 \times ^{-}6
bills	↓ ^{-}2 \times ^{-}6
for \$6 each.	↓ ^{-}2 \times ^{-}6

The postman remarks, "I hope you haven't worried about those bills; they're really for the lady upstairs. Give them back to me, and I'll run right upstairs with them this very minute."

As a result of the postman's visit, the housewife revises *upward* her estimate of available funds:

$$^{-}2 \times ^{-}6 = ^{+}12.$$

Considerable experience has convinced us that postman stories will work successfully for *you*, in *your* class, if you will take the trouble to work them out carefully and consistently.



CHAPTER 6

Postman Stories for Products

[page 23]

For products, like

$$^{-}2 \times ^{-}3,$$

Jerry makes up stories like this:

	<b>The postman</b>
Brings or takes away?	brings ↓ ^{-}2 \times ^{-}3
How many?	two ↓ ^{-}2 \times ^{-}3
Bills or checks?	checks ↓ ^{-}2 \times ^{-}3
For how much?	for \$3 each. ↓ ^{-}2 \times ^{-}3

ANSWERS AND COMMENTS

Can you make up a postman story for each problem?  
What answer do you get?

(1)  $+2 \times +3 =$

(2)  $+2 \times +5 =$

(3)  $+2 \times -3 =$

(4)  $+2 \times -5 =$

(5)  $+5 \times +7 =$

(6)  $-2 \times +1 =$

(1)  $+2 \times +3 = +6$  (Compare the illustrative example (i) in the introduction to this chapter.)

(2) The postman brings  
two  
checks  
for \$5 each.

$\downarrow$   
 $+2 \times +5 =$   
 $\downarrow$   
 $+2 \times +5 =$   
 $\downarrow$   
 $+2 \times +5 =$   
 $\downarrow$   
 $+2 \times +5 =$

As a result of *this* visit, Mrs. Housewife revises *upward*

$+2 \times +5 = +$   
↑

her estimate of available uncommitted money.  $+2 \times +5 = +10$   
↑

(3) The postman comes on Monday morning,

bringing  
two  
bills  
for \$3 each.

$\downarrow$   
 $+2 \times -3$   
 $\downarrow$   
 $+2 \times -3$   
 $\downarrow$   
 $+2 \times -3$   
 $\downarrow$   
 $+2 \times -3$

As a result of the postman's Monday-morning visit, Mrs. Housewife must revise *downward* her estimate of available spending money:

$+2 \times -3 = -6.$

(4) Postman brings two bills for \$5; Mrs. Housewife is thus "poorer" by \$10:

$+2 \times -5 = -10.$

(5) Postman brings 5 checks for \$7 each:

$+5 \times +7 = +35.$

(6) On Thursday morning, the postman pays a visit.

He takes away  
two  
checks  
for \$1 each.

$\downarrow$   
 $-2 \times +1$   
 $\downarrow$   
 $-2 \times +1$   
 $\downarrow$   
 $-2 \times +1$   
 $\downarrow$   
 $-2 \times +1$

As a result of the postman's Thursday-morning visit, the housewife must decrease her estimate of available funds (since she had, of course, previously included these two checks in her estimate of her available spending money). By how much? Evidently she must decrease the amount by \$2:

$$^{-}2 \times ^{+}1 = ^{-}2.$$

Notice that one might also make use of the identity

$$\square \times 1 = \square$$

in solving this problem.

(7)  $^{-}2 \times ^{+}5 =$

(7) Postman takes away two checks for \$5 each:

$$^{-}2 \times ^{+}5 = ^{-}10.$$

(8)  $^{-}2 \times ^{-}5 =$

(8) Postman takes away two bills for \$5 each. Since the housewife *had already allowed for these bills* in computing her estimate of available uncommitted funds, she may now breathe a sigh of relief and *increase* her estimate of uncommitted funds by \$10:

$$^{-}2 \times ^{-}5 = ^{+}10.$$

(9)  $^{-}2 \times ^{-}6 =$

(9) Similar to question 8 above:

$$^{-}2 \times ^{-}6 = ^{+}12.$$

(10)  $^{-}2 \times ^{-}6 =$

(10) Postman brings two bills for \$6 each. As a result of *this* visit by the postman, the housewife must decrease her estimate of her available "free" spending money:

$$^{-}2 \times ^{-}6 = ^{-}12.$$

(11)  $^{-}1 \times ^{-}1 =$

(11) Similar to question 8 above:

$$^{-}1 \times ^{-}1 = ^{+}1.$$

(12)  $^{-}5 + ^{+}3 =$

(12) *Note that this is a problem in addition.*

The postman brought

a check  $\downarrow$   $^{-}5 + ^{+}3$

for \$5  $\downarrow$   $^{-}5 + ^{+}3$

and he also brought

$\downarrow$   $^{-}5 + ^{+}3$

a check  $\downarrow$   $^{-}5 + ^{+}3$

for \$3.  $\downarrow$   $^{-}5 + ^{+}3$

As a result of *this* visit by the postman, the housewife will *increase* her estimate of available funds by \$8:

$$^{-}5 + ^{+}3 = ^{+}8.$$

(13)  $-^{-}5 =$

(13) This notation will *not* usually occur in our mathematical work. It is nice to know, however, that we *could* explain it with a postman story if we wanted to.

The postman took away	↓
	$-^{-}5$
a bill	↓
	$-^{-}5$
for \$5.	↓
	$-^{-}5$

The housewife accordingly increased her estimate of her uncommitted available funds by \$5:

$$-^{-}5 = ^{-}5.$$

In most mathematical situations we shall prefer to use the "oppositing" symbol  $^{\circ}$  to denote this unary operation, and to reserve the "subtraction" symbol  $-$  to denote a binary operation. Thus, instead of

$$-^{-}5 = ^{-}5,$$

we would usually write

$$^{\circ}(-5) = ^{-}5.$$

(14)  $^{-}3 \times ^{-}3 =$

(14) Similar to question 8 above:

$$^{-}3 \times ^{-}3 = ^{-}9.$$

(15)  $^{-}2 \times ^{-}2 =$

(15) Similar to question 8:

$$^{-}2 \times ^{-}2 = ^{-}4.$$

For questions 16, 17, and 18, the stories should cause no difficulty.

(16)  $^{-}3 \times ^{-}3 =$

(16)  $^{-}3 \times ^{-}3 = ^{-}9$

(17)  $^{-}2 \times ^{-}2 =$

(17)  $^{-}2 \times ^{-}2 = ^{-}4$

(18)  $^{-}2 \times ^{-}2 =$

(18)  $^{-}2 \times ^{-}2 = ^{-}4$

## KYE'S ARITHMETIC

This chapter reports an actual occurrence. While presenting subtraction, a third-grade teacher was discussing this problem:

$$\begin{array}{r} 64 \\ -28 \\ \hline \end{array}$$

She said, "I can't take eight from four, so I take ten from the sixty . . ."

At this point a third-grade boy, named Kye, interrupted and said, "Oh, yes you can! Four minus eight equals negative four . . ."

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \end{array}$$

"and sixty minus twenty equals forty . . ."

$$\begin{array}{r} 64 \\ -28 \\ \hline 4 \\ 40 \end{array}$$

"forty plus negative four equals thirty-six . . ."

$$\begin{array}{r} 64 \\ -28 \\ \hline -4 \\ \hline 40 \\ -4 \\ \hline 36 \end{array}$$

"so the answer is thirty-six."

This is as good an example of the difference between "traditional" programs and "modern" programs as we have encountered. The "traditional" teacher would presumably have said, "No, Kye, that's not the way you do it. Now you watch carefully while I do it the right way!"

The teacher in Kye's class, who was trying hard to catch the elusive spirit of "modern" mathematics, *actually listened to what Kye suggested, and actually thought about it.*

Hundreds of hours of tape-recorded classroom lessons have convinced us that children very often make up ingenious methods for solving problems, only to be overruled and "corrected" by a teacher who doesn't really understand the child's suggestion. (The author pleads guilty to this himself, and claims—in his defense—that children, although ingenious, are not clearly articulate; it is sometimes hard to figure out what a child means, particularly when the child's suggestion is quite unexpected!)

In any event, Kye's teacher *did* listen to his suggestion, she tried to understand and appreciate it, and she encouraged Kye (and the rest of the class) to explore Kye's method more fully. It turns out to be an excellent algorithm for subtraction—invented by a third-grade boy!

Suppose Kye's teacher had *rejected* Kye's suggestion. Kye would have been left with the feeling that "this stupid mathematics never does make sense; it never works out the way you'd think it would!"

Indeed, Kye would have been faced with the dilemma of persevering in his own vision of mathematical consistency at the price of severing diplomatic relations with the teacher or else hypocritically abandoning what he believed to be the true "pattern." Either way Kye would be the loser.

"Modern" mathematics teaching provides Kye with a third choice: to persevere in "figuring things out for himself" and to win approval—rather than rejection—for his creative efforts at making up his own methods for solving problems.

The mid-twentieth-century work on encouraging "divergent thinking" should never have been necessary. We ought never to have placed such ridiculously great emphasis on "convergent thinking!" Do we really *want* everyone to think in the same rigid mold?

Children *can* make up their own algorithms for arithmetic—if we adults give them a chance!

[If you wish to do further reading, see Davis (31) or *In-Service Course #1* (available from The Madison Project). You might also wish to view the Madison Project films "Kye's Arithmetic" and "Education Report: The New Math."]



## CHAPTER 7

## Kye's Arithmetic

[page 24]

(1) Miss Parsons was working this subtraction problem with her class:

$$\begin{array}{r} 64 \\ - 28 \\ \hline \end{array}$$

She said, "I can't take eight from four, so I'll regroup the sixty as ..." At this point a boy named Kye interrupted and said, "Oh, yes! Four minus eight is negative four

$$\begin{array}{r} 64 \\ - 28 \\ \hline -4 \end{array}$$

... and twenty from sixty is forty

$$\begin{array}{r} 64 \\ - 28 \\ \hline -4 \\ 40 \end{array}$$

... so that you get forty plus negative four, which is thirty-six."

$$\begin{array}{r} 64 \\ - 28 \\ \hline -4 \\ 40 \\ \hline 36 \end{array}$$

What would you say to Kye?

## ANSWERS AND COMMENTS

- (1) **If you explore this further (as we shall do in the next few pages) you will find that Kye's method is ingenious, valuable, and correct.**

(2) Can you use Kye's method on this problem?

$$\begin{array}{r} 83 \\ -25 \\ \hline \end{array}$$

(2)  $3 - 5 = \bar{2}$ :

$$\begin{array}{r} 83 \\ -25 \\ \hline \bar{2} \end{array}$$

$80 - 20 = 60$ :

$$\begin{array}{r} 83 \\ -25 \\ \hline \bar{2} \\ \hline 60 \end{array}$$

$60 + \bar{2} = 58$ :

$$\begin{array}{r} 83 \\ -25 \\ \hline \bar{2} \\ \hline 60 \\ \hline 58 \end{array}$$

(3) Some other students extended Kye's method. They decided to write "negative signs" over the digits to which they apply, so that

$$5\bar{3}$$

means

$$50 - 3$$

or

$$50 + \bar{3}$$

If  $3\bar{4}$  means "thirty plus four," can you say what each of these numerals means?

- (a) 72
- (b)  $7\bar{3}$
- (c)  $1\bar{3}$
- (d) 21
- (e) 45
- (f) 55

- (3) (a)  $70 + 2$
- (b)  $70 - 3$ , or (if you prefer)  $70 + \bar{3}$
- (c)  $10 - 3$ , or  $10 + \bar{3}$
- (d)  $20 + 1$
- (e)  $40 + 5$
- (f)  $50 + 5$

Note that this use of "negative digits" offers an unexcelled opportunity to study place-value numerals.

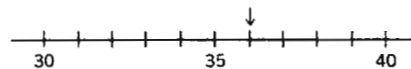
[page 25]

(4) Cynthia wrote:

$$\begin{array}{r} 64 \\ -28 \\ \hline 44 \end{array}$$

What do you think ?

(4) Cynthia's "answer" is perfectly acceptable. Its meaning is clear (compare question 3 above), and one can stop with "44" as the "answer." Alternatively, one can convert to our standard numerals:



$$4\bar{4} = 40 - 4 = 30 + 6 = 36$$

(5) Can you work out these problems by two (or more) different methods?

$$(a) \begin{array}{r} 23 \\ + 13 \\ \hline \end{array}$$

$$(b) \begin{array}{r} 59 \\ + 37 \\ \hline \end{array}$$

(5) (a) Method 1

$$3 + \overline{3} = 0: \quad \begin{array}{r} 23 \\ \overline{13} \\ 0 \end{array}$$

$$20 + 10 = 30: \quad \begin{array}{r} 23 \\ + \overline{13} \\ 0 \\ \hline 30 \end{array}$$

$$30 + 0 = 30: \quad \begin{array}{r} 23 \\ + \overline{13} \\ 0 \\ \hline 30 \\ \hline 30 \end{array}$$

Method 2 (converting to standard numerals)

$$\overline{13} = 10 - 3 = 7: \quad \begin{array}{r} 23 \\ + 7 \end{array}$$

$$23 + 7 = 30: \quad \begin{array}{r} 23 \\ + 7 \\ \hline 30 \end{array}$$

(b) Method 1

$$9 + \overline{-7} = 2: \quad \begin{array}{r} 59 \\ + \overline{37} \\ 2 \end{array}$$

$$50 + 30 = 80: \quad \begin{array}{r} 59 \\ + \overline{37} \\ 2 \\ \hline 80 \end{array}$$

$$80 + 2 = 82: \quad \begin{array}{r} 59 \\ \overline{37} \\ 2 \\ \hline 80 \\ \hline 82 \end{array}$$

Method 2 (converting to standard numerals)

$$\overline{37} = 30 - 7 = 20 + 3 = 23: \quad \begin{array}{r} 59 \\ + 23 \end{array}$$

Now proceed as usual ...

(6) Some other students made up a method for subtracting. If we use their method, the problem

$$\begin{array}{r} 64 \\ - 28 \\ \hline \end{array}$$

means "how far (on the number line) is it from 28 to 64?"

(6) This method is quite satisfactory, and one more example of the ingenuity of children when we appreciate their ingenuity instead of crushing it.

This method was made up by some sixth-grade children in a suburb of Seattle, Washington.



We'll see:

$$\begin{array}{r} 28 \text{ plus } 2 \text{ gets you to } 30 \\ 64 \\ -28 \\ \hline 2 \end{array}$$

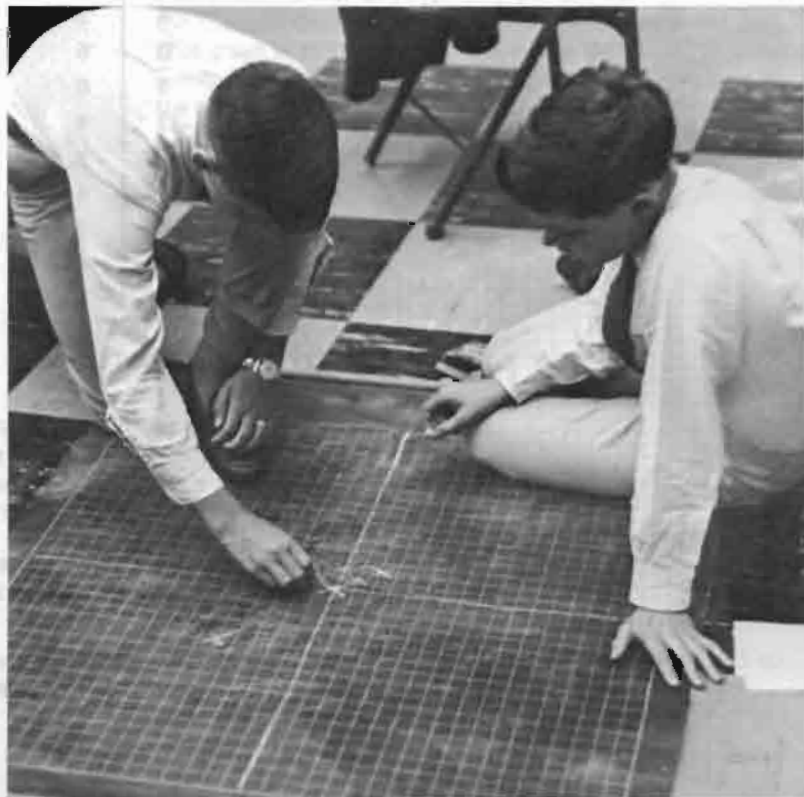
$$\begin{array}{r} \dots \text{ plus } 30 \text{ gets you to } 60 \\ 64 \\ -28 \\ \hline 2 \\ 30 \end{array}$$

$$\begin{array}{r} \dots \text{ plus } 4 \text{ gets you to } 64 \\ 64 \\ -28 \\ \hline 2 \\ 30 \\ \hline 4 \end{array}$$

$$\begin{array}{r} \dots \text{ and altogether we've added} \\ 2 \text{ and } 30 \text{ and } 4, \\ 64 \\ -28 \\ \hline 2 \\ 30 \\ \hline 4 \\ \hline 36 \end{array}$$

... which is 36.

What do you think about this method?



## GRAPHS WITH SIGNED NUMBERS

If you have not already done so, this would be a good time to view the film "Postman Stories." You may also be interested in the film "Circles and Parabolas."



### CHAPTER 8

## Graphs With Signed Numbers

[page 26]

- (1) Can you show the truth set for

$$(\square \times \square) + (\triangle \times \triangle) = 25,$$

by means of a table and a graph? (Use both positive and negative numbers.)

## ANSWERS AND COMMENTS

- (1) It is probably wisest to restrict ourselves to *integers*. If we do, we shall find exactly 12 pairs of numbers that will produce a true statement; consequently, the graph will show exactly 12 points.

$\square$	$\triangle$
0	5
5	0
+3	+4
+4	+3
0	-5
-5	0
+3	-4
+4	-3
-3	+4
-4	+3
-3	-4
-4	-3

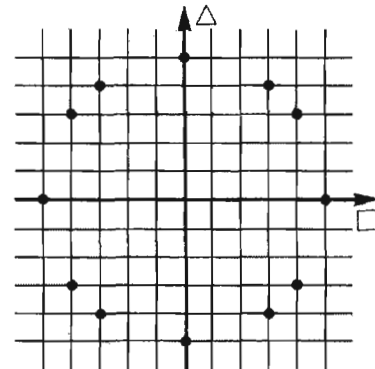


Table for Truth Set

Graph for Truth Set

Actually, as the film "Postman Stories" shows very well, this is a case where the algebra aids the geometry, and the geometry aids the algebra. Once you have located a few points, both geometric and algebraic calculations work together to help you locate additional points.

For the continuous case, where we include *rational* numbers, compare *Discovery* (Teachers' Text), page 199.

- (2) Can you show the truth set for

$$\square \times \triangle = 36,$$

by means of a table and a graph?

- (2) We might, again, use *only integers*. If we make this choice, the table and graph look like the following.

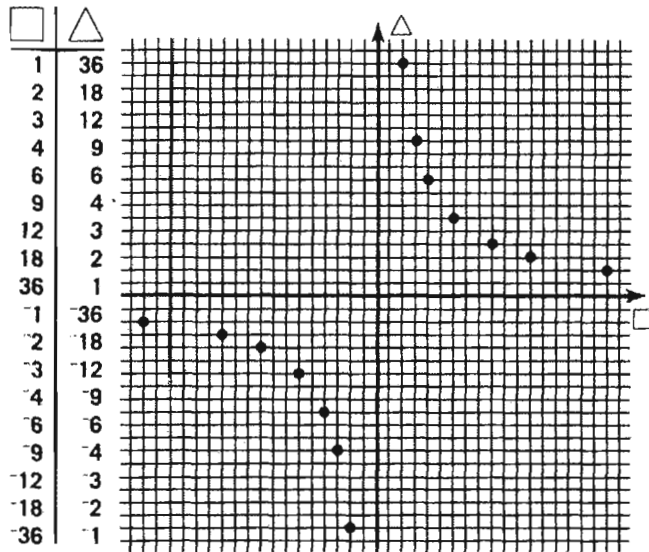
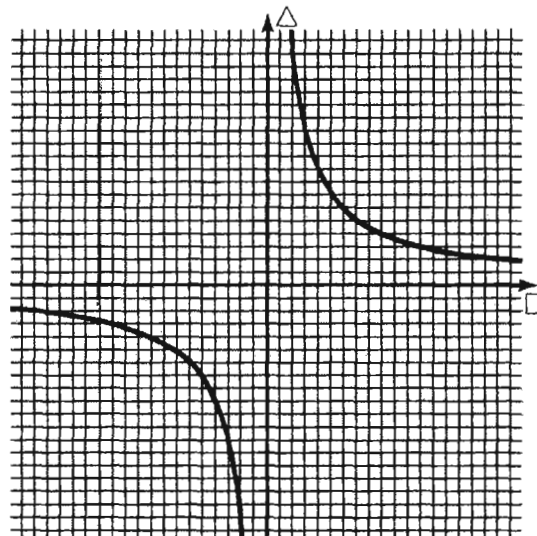


Table for Truth Set

Graph for Truth Set

There are many elements of the truth set that involve fractions; indeed, if we allow fractional solutions, we get what appears to be a continuous, smooth curve, instead of merely isolated points. (This curve is known as a hyperbola, and was used by the architect Gyo Obata in designing the planetarium in St. Louis, Missouri. A photograph of this planetarium appears in the *Student Discussion Guide*. Of course, the perspective of the photo distorts somewhat the actual curve.)

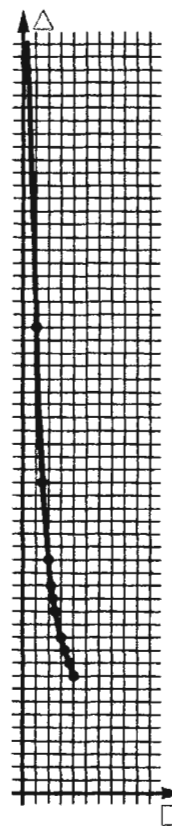
The following shows what the curve might look like if we allow the use of fractions.



Or, we might look more closely at the following which shows a small piece of this curve, with a portion of the table.

□	△
1	36
1½	24
2	18
2¼	16
2⅔	15
2⅚	14
3	12
3⅓	11
3⅔	10
4	9
⋮	⋮
⋮	⋮

Table for Truth Set  
(if we agree to use fractions)



Graph for Truth Set

(3) Can you show the truth set for

$$\square + \triangle = 10,$$

by means of a table and a graph? (Use positive numbers, negative numbers, and fractions.)

As illustrated at the right,\* a piece of the graph of the truth set of problem 2,

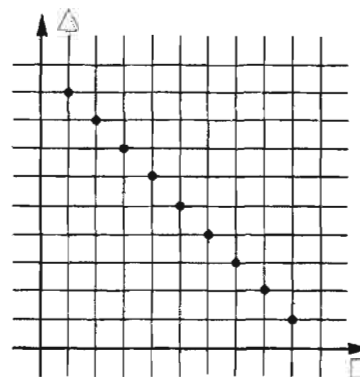
$$\square \times \triangle = 36.$$

was used by architect Gyo Obata in designing the planetarium in St. Louis, Missouri. A continuous, smooth curve (a hyperbola) was obtained by allowing fractional solutions.

(3) If we use only positive whole numbers, the truth set looks like the following.

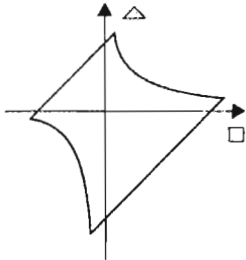
□	△
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1

Table for Truth Set  
(using only positive whole numbers)

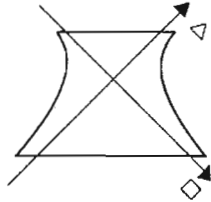


Graph for Truth Set  
(using only positive whole numbers)

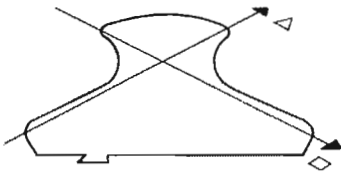
\*On page 95 of the Teachers' Text



The picture is rotated so that the  $\square$  and  $\triangle$  axes appear in their usual positions.



In this diagram the perspective is changed so that the  $\square$  and  $\triangle$  axes appear perpendicular.



Because of perspective and changes of scale, the  $\square$  and  $\triangle$  axes that fit the profile curve do not seem to be perpendicular.



Actual view of the St. Louis planetarium as seen from the ground.

If, instead, we agree to use *integers* (i.e., the replacement set for each variable shall be  $\{\dots, 5, 4, 3, 2, 1, 0, 1, 2, 3, \dots\}$ ), then the table and graph look like the following.

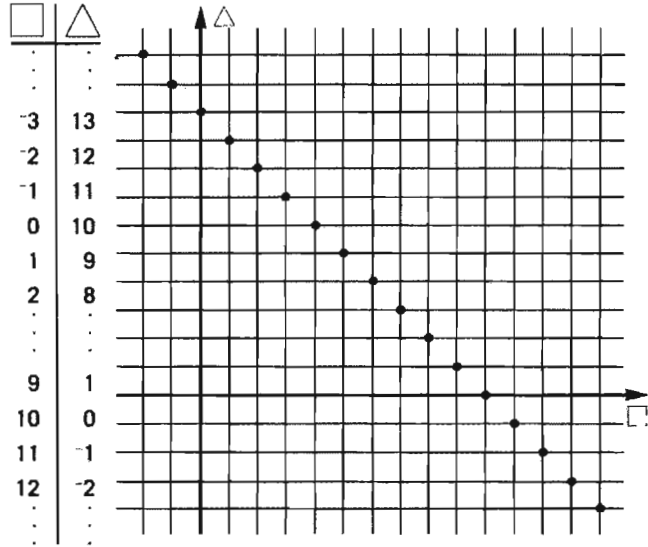
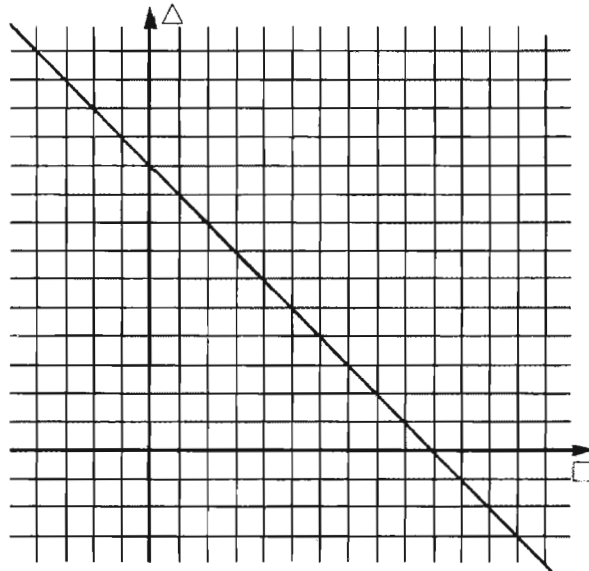


Table for Truth Set (if we agree to use integers)

Graph for Truth Set (if we agree to use integers)

Finally, if we use also *fractions* (positive and negative), the graph looks like the following.



Graph for Truth Set (if we agree to use integers and fractions)

(4) Can you show the truth set for

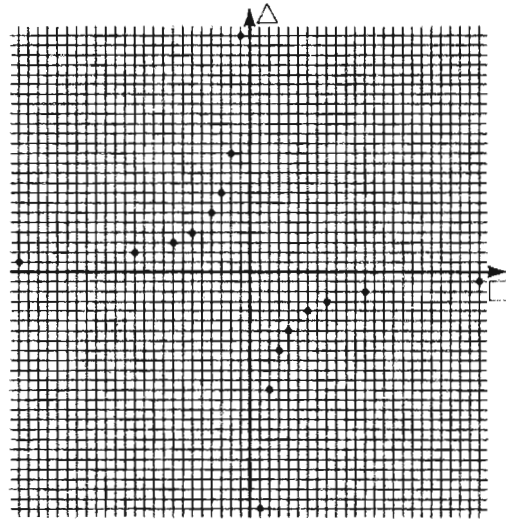
$$\square \times \triangle = -24,$$

by means of a table and a graph?

[page 27]

(4) If we use only *integers*, the table and graph look like the following.

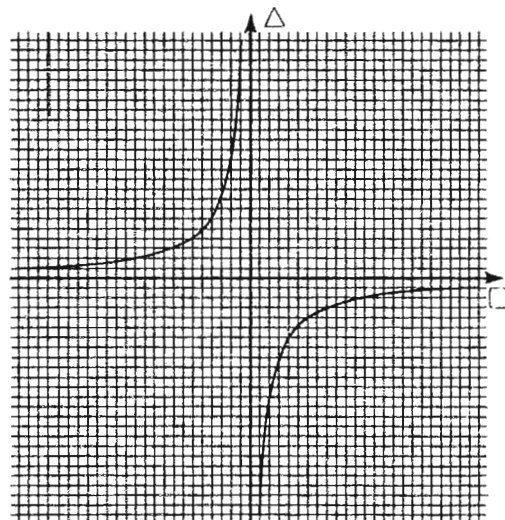
$\square$	$\triangle$
-1	24
-2	12
-3	8
-4	6
-6	4
-8	3
-12	2
-24	1
1	-24
2	-12
3	-8
4	-6
6	-4
8	-3
12	-2
24	-1



Graph for Truth Set (using only integers)

Table for Truth Set (using only integers)

If we agree to allow the use of *fractions* (positive and negative), the graph becomes another example of a hyperbola.

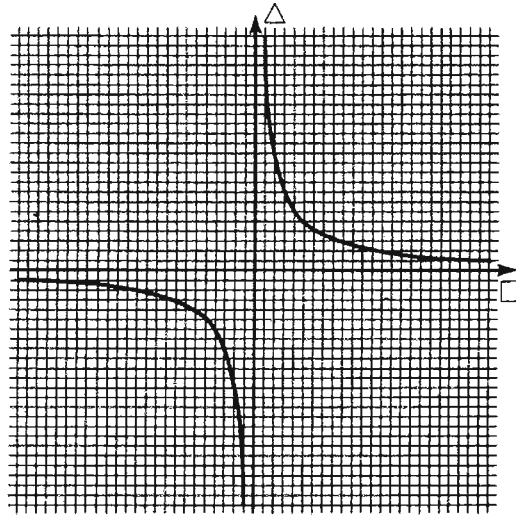


Graph of Truth Set (if we agree to allow the use of fractions)

Can you make a graph for each truth set?

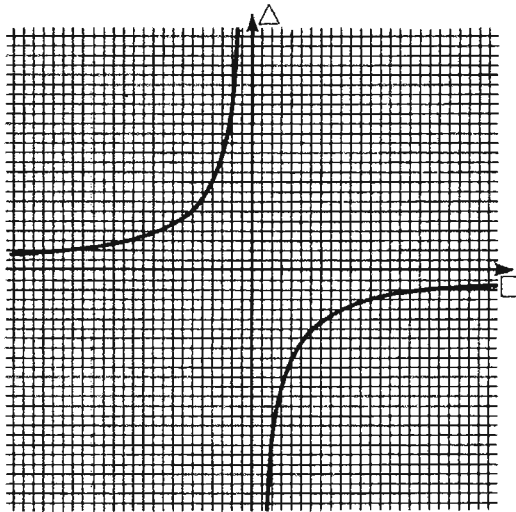
(5)  $\square \times \triangle = '24$

(5)



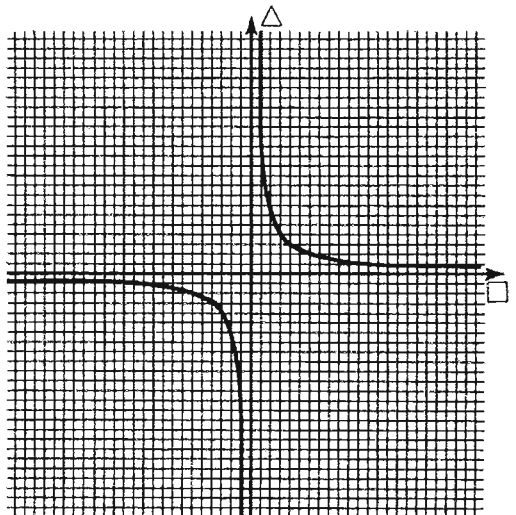
(6)  $\square \times \triangle = '36$

(6)

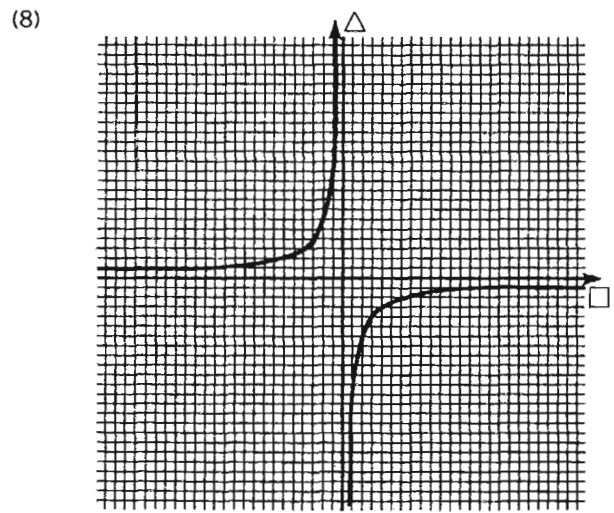


(7)  $\square \times \triangle = '12$

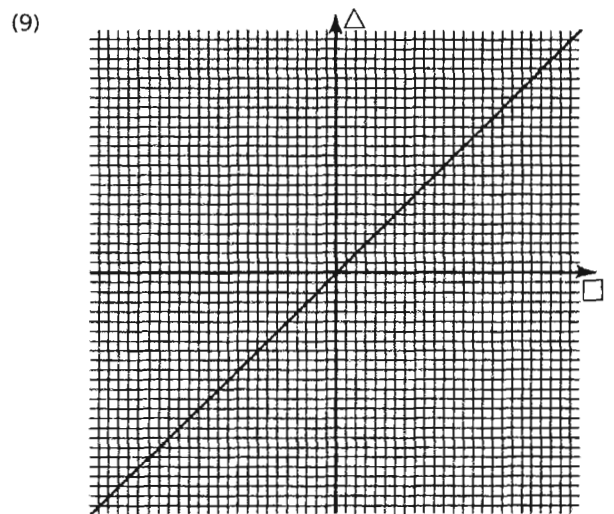
(7)



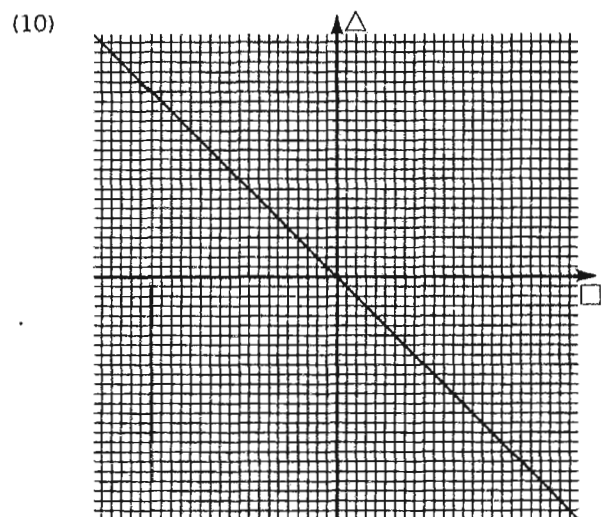
(8)  $\square \times \triangle = -12$



(9)  $\square - \triangle = 0$



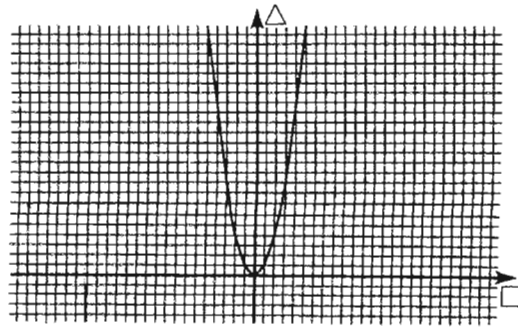
(10)  $\square + \triangle = 0$





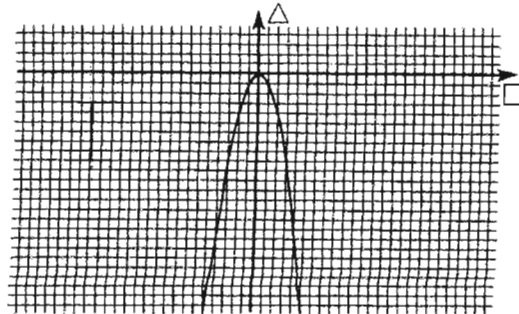
(11)  $\triangle = \square \times \square$

(11)



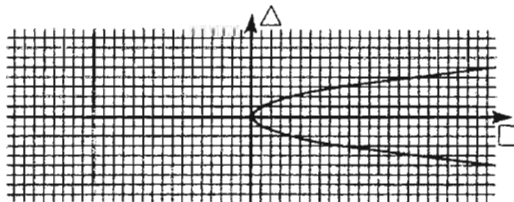
(12)  $\triangle + (\square \times \square) = 0$

(12)



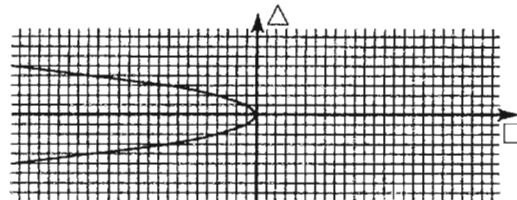
(13)  $\square = \triangle \times \triangle$

(13)



(14)  $\square + (\triangle \times \triangle) = 0$

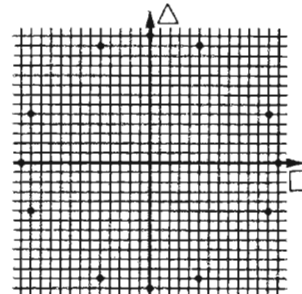
(14)



(15)  $(\square \times \square) + (\triangle \times \triangle) = 169$

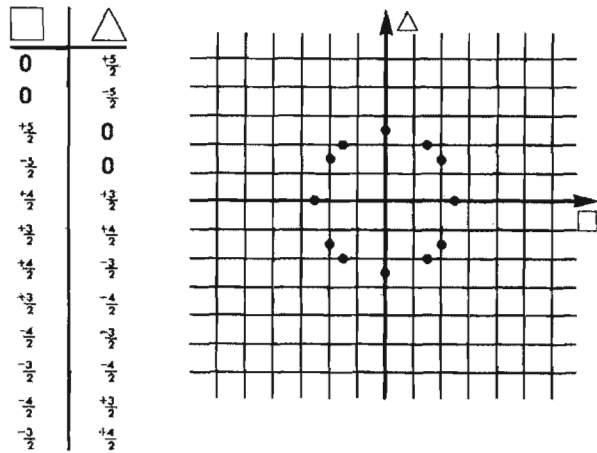
(15) Using integers only:

$\square$	$\triangle$
-13	0
-12	-5
+5	+12
0	+13
-13	0
-12	+5
-5	-12
0	-13
-12	-5
-5	-12
+12	-5
+5	-12



(16)  $(\square \times \square) + (\triangle \times \triangle) = \frac{32}{7}$

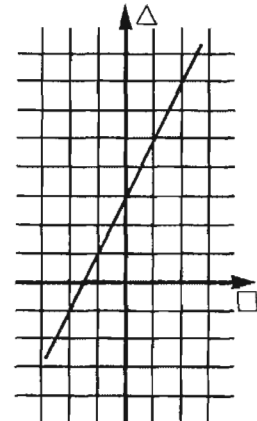
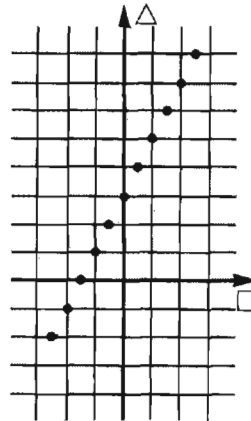
(16) Using only fractions (positive and negative) with denominators equal to two:



(17)  $(\square \times '2) + '3 = \triangle$

(17) Using only integers and fractions with denominators of 2:

Using all fractions, positive and negative:

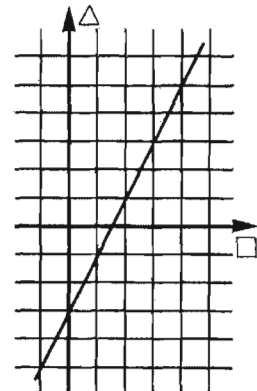
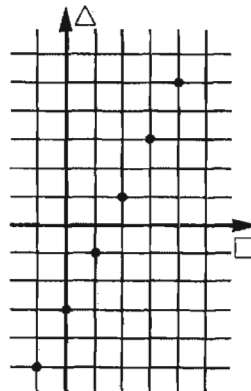


Compare *Discovery*, Chapters 11, 15, 17, 18, and 35.

(18)  $(\square \times '2) + '3 = \triangle$

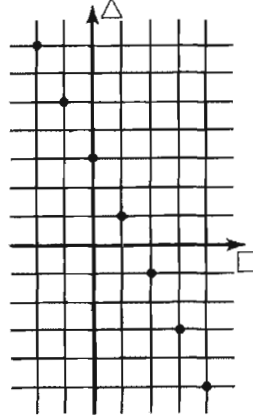
(18) Using integers only:

Using fractions:

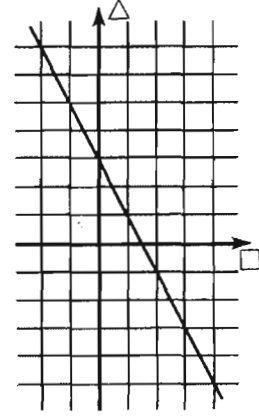


(19)  $(\square \times -2) + '3 = \triangle$

(19) Using integers only:



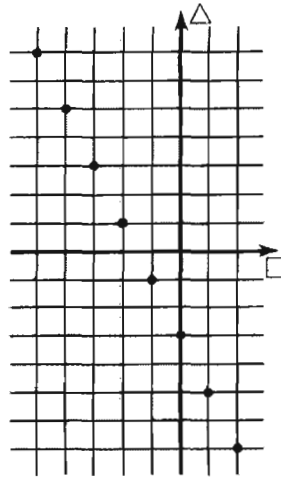
Using fractions:



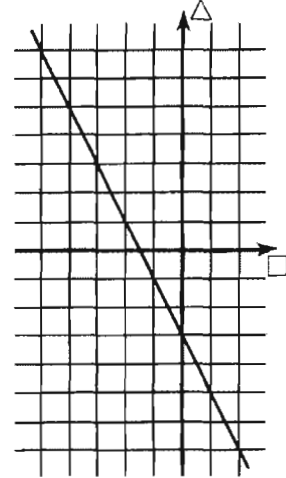
Compare *Discovery*, Chapters 11, 15, 17, 18, and 35.

(20)  $(\square \times -2) + '3 = \triangle$

(20) Using integers only:



Using fractions:



(21) Widge says she uses the symbol "°" like this:

- $^{\circ}(+3) = -3$
- $^{\circ}(-3) = +3$
- $^{\circ}(+10) = -10$
- $^{\circ}(+ \frac{1}{2}) = - \frac{1}{2}$
- $^{\circ}(+1.1) = -1.1$

(21)  $^{\circ}(+5) = -5$ . This is read either as "the opposite of positive five is negative five" or as "the additive inverse of positive five is negative five."

You may wish to view the film "Second Lesson."

What would this be?

$^{\circ}(+5) =$

(22) Can you find these “opposites”?

(a)  $\overset{\circ}{-}('1) =$

(b)  $\overset{\circ}{-}(-4) =$

(c)  $\overset{\circ}{-}(\frac{1}{10}) =$

(d)  $\overset{\circ}{-}(0) =$

[page 28]

(23) Lex says that Widge is finding additive inverses and that she finds  $\overset{\circ}{-}('1)$  by asking, “What can I add to positive one to end up with zero?” What do you think?

(24) Can you find the truth set for each open sentence?

(a)  $'5 + \square = 0$

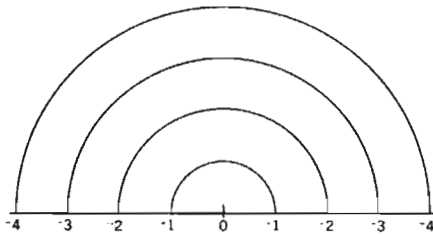
(b)  $'7 + \square = 0$

(c)  $'3 + \square = 0$

(d)  $0 + \square = 0$

Can you give the truth set of each of these open sentences another name?

(25) Cynthia made a rainbow picture:



She says that you find the additive inverse of a number by “going to the opposite end of the rainbow.”

What do you think?

(26) Can you find the additive inverse of each number?

(a)  $'10$

(b)  $'15$

(c)  $'3$

(d)  $0$

(22) (a)  $\overset{\circ}{-}('1) = '1$ . Read either as “the opposite of positive one is negative one” or as “the additive inverse of positive one is negative one.”

(b)  $\overset{\circ}{-}(-4) = '4$

(c)  $\overset{\circ}{-}(\frac{1}{10}) = '10$

(d)  $\overset{\circ}{-}(0) = 0$

(23) Lex is right. We are actually using the identity

$$\square + \overset{\circ}{-}\square = 0.$$

For an explanation of what we mean by *identities*, see Chapter 18 of *Explorations* or Chapter 5 of *Discovery*.

(24) (a)  $\{-5\}$ .  $'5$  is the additive inverse of  $'5$ .

(b)  $\{-7\}$ .  $'7$  is the additive inverse of  $'7$ .

(c)  $\{-3\}$ .  $'3$  is the additive inverse of  $'3$ .

(d)  $\{0\}$ .  $0$  is the additive inverse of  $0$ .

(25) Cynthia is right.

(26) (a)  $'10$

(b)  $'15$

(c)  $'3$

(d)  $0$

## USING NAMES AND VARIABLES IN MATHEMATICS

This chapter is intended to be somewhat more of a "reading" chapter, although some valuable class discussion might occur after the students have tried to read the chapter and to understand it by themselves.

Mathematically, the chapter is intended to introduce these ideas:

- (i) the distinction between "names" and "things";
- (ii) the meaning of the symbol =, or of the idea of "is equal to," or "equality";
- (iii) the logical operation known as PN;
- (iv) the mathematical operation known as UV.

Let us review these matters very briefly right now. First, it is sometimes desirable to distinguish some *thing* from some *name* for that thing. [This distinction has been made most forcefully and most lucidly by Vaughan and Beberman in the UISCM materials, see Appendix A: Beberman (87).]

Consider the example of milk. If I have the *thing* milk on my paper, that is a mess, probably attributable to my two-year-old daughter. But if I have on my paper a *symbol* or a *name* to refer to milk, that is an entirely different matter. Indeed, a *symbol* for milk has been used four times in this paragraph.

Here is a second example. The statement

Mary has four letters

might, without further clarification, mean either that the *girl* Mary has received four letters in her mail box this morning, or it may mean that the name *Mary* has the four letters *M*, *a*, *r*, and *y*.

Suppose we wish to name the number *seven*. We could write

7    or    VII    or    seven.

We could also write

6 + 1    or    5 + 2

or 1 + 1 + 1 + 1 + 1 + 1 + 1

or 10 - 3    or    35 ÷ 5.

Any of these would be *names* for the *number seven*.

This may not seem very exciting or very important. But wait until we come to *use* this idea in our efforts to work out the meaning for the sign =, and for the logical operation that we shall call PN.

Second, what shall we mean by the symbol =? You will hear people, when they are speaking carelessly, say that "some number" is equal to "some other number." But—as we ask in the *Student Discussion Guide*—*how* can some number be *equal* to some *other* number? This does not make sense. Nor is it the approach of modern logicians. They have preferred to interpret the statement  $1 + 1 = 2$  to mean

"1 + 1" names some number, and "2" names some number, and – in fact – they both name the *same* number.

Thus, equality is primarily a statement about *names* and not about *things*. We shall carry this same interpretation along, whether the "things" be numbers, statements, matrices, geometrical entities, or what have you.

In particular, we shall name *matrices* by writing arrays of numbers such as

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 5 \\ 2 \\ 25 \end{pmatrix}$$

We shall say that two such names will *name the same matrix* if and only if they "look" as if they do. That is

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

*names the same matrix as*

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

but the name

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

does *not* name this same matrix. We shall look into this matter when we begin our study of matrices.

Third, we have a logical operation which we shall refer to as the principle of names and shall abbreviate as PN. The meaning of PN is stated at moderate length in the *Student Discussion Guide*. We shall try to understand what it is all about when we come to this part of the student chapter.

Finally, the use of variables (or UV) is not something new. It is our old friend from Chapter 1, reviewed here in order to compare it with PN, which is *somewhat* like it.



## CHAPTER 9

## Using Names and Variables in Mathematics

[page 29]

### THE MEANING OF "EQUAL"

How can "two different numbers" ever be equal? This question poses the kind of problem we often encounter when we think carefully about the words we use.

## ANSWERS AND COMMENTS

In mathematics it is sometimes important to use words rather carefully. Mathematicians and logicians want to avoid contradictions in what they say. Consequently, they have thought a good deal about the problem of "two different numbers" being "equal," and have decided to resolve the difficulty this way:

(a) Mathematicians distinguish "symbols" or "names" from "things" or "ideas." You have an idea of two; but you cannot write the idea on the chalkboard. (They don't sell that kind of chalk!)

What you write on the board is a symbol or a name, such as

2

or

II

or

two.

(b) Mathematicians agree that "equality," which they write by using the symbol "=", is a statement about names, and not a statement about things or ideas.

Thus, "two different numbers" never will be "equal." When we write

$$1 + 1 = 2,$$

what we shall mean by this is that

$$1 + 1$$

is a name and that

2

is a name, and, in fact, these names both name the same "thing" or "idea." Both

$$1 + 1$$

and

2

are names for the number two.

(1) In Paul's class, there is a girl named Sandy Davis. Paul claims that we would be using the symbol "=" correctly, as it is used by modern mathematicians, if we wrote

$$\text{Sandy} = \text{Miss Davis.}$$

Do you agree?

[page 30]

(2) Jill says we could also write

Hawaii = the only state consisting  
entirely of islands.

Do you agree?

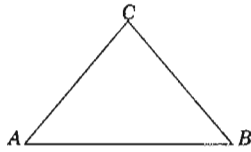
(1) **Paul is correct. When we write**

$$\text{Sandy} = \text{Miss Davis,}$$

**we are saying that "Sandy" is a name and "Miss Davis" is a name and that both name the same person.**

(2) **This usage of the = sign would be correct, in the light of modern logic and mathematics.**

- (3) In the triangle  $ABC$ ,



can you identify which angle is meant by the notation  $\angle CAB$ ? Which angle do we mean when we write

$$\angle CBA?$$

(4) George says that the triangle  $ABC$  has some kind of symmetry. In fact, he took a protractor and measured  $\angle CAB$  and also  $\angle CBA$ . He concluded that they were equal, and so he wrote

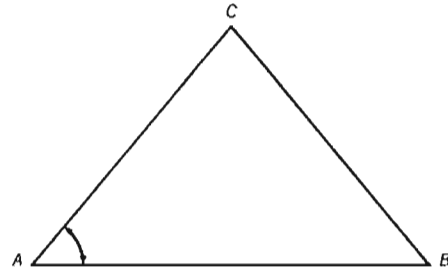
$$\angle CAB = \angle CBA.$$

What do you think?

- (3) The notation

$$\angle CAB$$

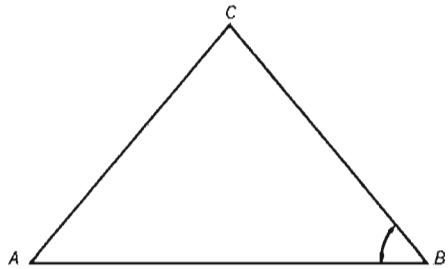
refers to this angle:



whereas the notation

$$\angle CBA$$

refers to this angle:



- (4) George is correct insofar as if we put a protractor on these two angles, we shall get the same reading in both cases. However, when George writes

$$\angle CAB = \angle CBA,$$

he goes too far! According to *modern* usage, the statement

$$\angle CAB = \angle CBA$$

would say:

“ $\angle CAB$ ” names something, and “ $\angle CBA$ ” names something, and, in fact, they name the *same* thing.

However, “ $\angle CAB$ ” does *not* name the same angle that “ $\angle CBA$ ” names, as our pictures in question 3 clearly show.

Please notice that this is a *new* usage of “equals.” When I was in school—and possibly when you were—one learned that “the base angles of an isosceles triangle are equal.” This is using the idea of equality precisely as George did when he wrote

$$\angle CAB = \angle CBA.$$

Both George and my past education are wrong, if one judges in terms of *today's* use of the symbol  $=$ , or the idea of equality. For an excellent discussion of this matter, see Appendix A, Moise (94).



(4) Can you make up some statements where you use the symbol "=" the same way that modern mathematicians do? Can you explain the meaning of your statements?

### THE "PRINCIPLE OF NAMES"

Mathematicians have a rule, which we shall call the "principle of names," and will abbreviate "PN."

The rule PN says roughly this:

If you take any true statement, take some occurrence of some name in this statement, erase it, and replace it by another name for the same thing, then the new statement you get will also be true.

The same thing holds for false statements. If you start with a false statement and replace some occurrence of some name by another name for the same thing, then the new statement will also be false.

To understand PN, let's look at some examples.

(6) Let's use Sandy Davis's name again. If we start with the statement

Sandy was born in St. Louis, Missouri,

(which is true) and if we erase the name "Sandy"

■■■■ was born in St. Louis, Missouri,

and replace it by "Miss Davis" (which is another name for the same person), we get the statement

Miss Davis was born in St. Louis, Missouri.

Now, according to PN, the statement [page 31]

Miss Davis was born in St. Louis, Missouri, should also be true. Is it?

(7) Suppose that we start with the statement

Accra is a city very near the equator,

and suppose that we say

Accra is the capital of Ghana.

Can we write

The capital of Ghana is a city very near the equator?

(8) Can we start with the statement

$$2 + 4 = 6$$

and use the fact that

$$4$$

names the same number that

$$3 + 1$$

does, to write

$$2 + (3 + 1) = 6?$$

(5) Here are a few:

$$2 \times 2 = 4$$

This says that " $2 \times 2$ " names some number, and "4" names some number, and—in fact—they both name the same number.

$$\text{Deutsch} = \text{German}$$

This says that "Deutsch" is a name of a language, and "German" is the name of a language, and—in fact—they both name the same language.

$$\text{VII} = \text{seven}$$

This says that "VII" names a number, and "seven" names a number, and—in fact—they both name the same number.

(6) Yes. This is a correct use of PN.

(7) Yes. This is a correct use of PN.

(8) Yes. This also is a correct use of PN.

(9) Can we start with the statement

$$5 + 5 + 5 = 15$$

and use the fact that

$$5$$

names the same number that

$$3 + 2$$

does, to write

$$5 + 5 + (3 + 2) = 15?$$

Is this a correct use of PN?

**A METHOD FOR SHOWING WHERE (AND HOW) WE HAVE USED PN**

Sometimes mathematics looks complicated when you see it written down, and it is useful to have ways of "writing notes to ourselves" so that we can keep track of what is going on. This occasionally happens when we are using PN. In order to keep track of where we use PN, we can use either of two methods. For one method we mark a "gaping hole" for the name we "erase," and into the "gaping hole" we place the new name for the same thing. For the other method we agree to underline with a heavy black line the name which we "erase" and also the new name for the same thing.

**Example 1**

The "gaping-hole" method of writing:

(i) Sandy was born in St. Louis.

We erase the name "Sandy."

(ii) ██████ was born in St. Louis.

Into the "gaping hole," we insert the name "Miss Davis."

(iii) Miss Davis was born in St. Louis.

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The "underlining" method of writing:

(i) Sandy was born in St. Louis.

(ii) Sandy = Miss Davis.

(iii) Miss Davis was born in St. Louis. PN from line (i), using line (ii).

Notice that, in the line above, we have given an "explanation" of what we did, by writing

PN from line (i), using line (ii).

**Example 2**

The "gaping hole" method of writing:

(i)  $(3 \times 6) + (3 \times 1) = 21$

(9) Yes, it is. Please notice that, in using PN, we may delete one occurrence of the name "5"

$$5 + 5 + \cancel{5} = 15$$

to get

$$5 + 5 + \blacksquare = 15.$$

↑  
gaping hole

Into this gaping hole, we put another name for the same thing (namely, "3 + 2") to get

$$5 + 5 + (3 + 2) = 15.$$

We did not have to do the same thing for all the other occurrences of 5 in this statement. In this respect, PN is quite different from UV.

Note: In questions 6 through 9 we started with true statements. We then used PN. Since PN does not change the truth value of a statement, we ended up with true statements.

If, instead, we start with a false statement, we shall end up with a false statement. Here is an example:

New York State is south of the Mason-Dixon line.

Now, "New York State" is also called "the Empire State." Hence, we could use PN, erase the name "New York State":

████████████████████ is south of the Mason-Dixon line.  
↑  
gaping hole

And then put in another name for the same thing:

The Empire State is south of the Mason-Dixon line.

This last statement is false, and so is the statement we started with.

(ii) Now, " $3 \times 1$ " names the same thing that "3" names.

(iii) Hence, we can "erase" the name " $3 \times 1$ ":

$$(3 \times 6) + \text{ } = 21.$$

(iv) Into the "gaping hole" we can put "3," to get

$$(3 \times 6) + 3 = 21.$$

The "underlining" method of writing:

(i)  $(3 \times 6) + \underline{(3 \times 1)} = 21$

(ii)  $3 \times 1 = 3$

(iii)  $(3 \times 6) + \underline{3} = 21$       PN from line (i),  
using line (ii).

(10) Try to rewrite your work on questions 8 and 9, using the "underlining" notation and "explaining" the final step, as in the preceding examples.

#### THE "USE OF VARIABLES "

There is another rule in mathematics which looks somewhat like PN but is really quite different. We want to be careful not to get the two mixed up.

This other rule is the rule for using variables, which we shall abbreviate "UV."

We have actually learned about UV in Chapter 1, but we did not name it. Let's look at a few examples.

(11) If you start with the open sentence

$$\square + \square = 2 \times \square$$

and if you make a numerical replacement for the variable like this

$$\text{UV: } 3 \rightarrow \square,$$

what statement do you get?

[page 33]

(12) If you start with the open sentence

$$\square + 1 < 5$$

and if you make a numerical replacement for the variable like this

$$\text{UV: } 8 \rightarrow \square,$$

what statement do you get?

(13) If you start with the open sentence

$$(\square + \square) + \square = 3 \times \square$$

and if you use the fact that the "open name"

$$\square + \square$$

will always name the same thing that

$$2 \times \square$$

names, can you therefore write

$$(2 \times \square) + \square = 3 \times \square?$$

Have you used UV or PN? Did you use it correctly?

(10) Here is question 8, written using the "underlining" notation:

(i)  $2 + \underline{4} = 6$

(ii)  $4 = 3 + 1$

(iii)  $2 + \underline{(3 + 1)} = 6$       PN from line (i), using line (ii).

Here is question 9, written using the "underlining" notation:

(i)  $5 + 5 + \underline{5} = 15$

(ii)  $5 = 3 + 2$

(iii)  $5 + 5 + \underline{(3 + 2)} = 15$       PN from line (i), using line (ii).

(11)  $3 + 3 = 2 \times 3$

Remember, in using UV, you must make the same replacement for every occurrence of the variable. That is, if you put 3 into the first  $\square$ , you must put 3 into every  $\square$ . (Contrast this with question 9.)

(12)  $8 + 1 < 5$ . Notice that one cannot tell anything about the truth or falsity of an open sentence. In this case, using 8 as the numerical replacement results in a false statement. If the numerical replacement had been 2, UV:  $2 \rightarrow \square$ , the resulting statement,  $2 + 1 < 5$ , would be true.

(13) This is a correct use of PN.

- (14) If you start with the open sentence

$$(\square + \square) + \square = 3 \times \square$$

and if you make a numerical replacement for the variable by "putting 4 in every  $\square$ ," what statement do you get? Did you use UV or PN?

- (15) Al started with the open sentence

$$(\square + \square) + \square = 3 \times \square,$$

and "put 5 into all the  $\square$ 's on the left side of the = sign." He got, as a result, the open sentence

$$(5 + 5) + 5 = 3 \times \square.$$

Was Al using UV or PN? Did he use it correctly?

- (16) If you start with the open sentence

$$(2 \times \square) + \square = 3 \times \square$$

and use the fact that the open name

$\square$

will always name the same thing that

$$1 \times \square$$

names, can you therefore write

$$(2 \times \square) + \cancel{\square} = 3 \times \square$$

(Erase this ...)

$$(2 \times \square) + \underbrace{(1 \times \square)}_{\uparrow} = 3 \times \square?$$

(and put this in its place.)

Did you use UV or PN? Did you use it correctly?

- (17) Try to write out your work for question 16, using the "underlining" notation.

[page 34]

- (18) How is UV different from PN?

- (14)
- $(4 + 4) + 4 = 3 \times 4$
- . We used UV.

- (15) Al was apparently *trying* to use UV, but he did not use it correctly. He put 5 into the first three  $\square$ 's, but he didn't put 5 into the last  $\square$ . (Compare questions 9 and 11 above.)

- (16) This is a correct use of PN.

$$(17) (2 \times \square) + \underline{\square} = 3 \times \square$$

$$\square = 1 \times \square$$

$$(2 \times \square) + \underline{(1 \times \square)} = 3 \times \square$$

- (18) This is a little bit like asking "What's the difference between a United States mail box and a 1964 Chevrolet?" There are lots of differences. One that should be noted carefully is that UV requires us to look at what we do to the first  $\square$ , and to treat every other  $\square$  exactly the same way; PN makes no such requirement (compare question 9).

Also, UV is possible only if we have an *open sentence* that contains at least one variable (whether we write the variable as  $\square$  or  $\triangle$  or  $x$  or  $A$  does not matter). On the other hand, PN does not necessarily involve variables.

A correct use of PN will never change the truth value of a statement. A correct use of UV will often change truth values. For example,

$$3 + \square = 5$$

is *open*, whereas, if we use UV,

$$\text{UV: } 11 \rightarrow \square,$$

we get the statement

$$3 + 11 = 5,$$

which is *false*.

(19) What number is named most often below?

seven

one-half

21

$5 - 2$

$X - III$

$\frac{1}{2}$

1,000,000,000,003

11-4

VII

0.0004

SIX

35

$(1 + 2) + 4$

(19) The number seven is named six times, if we agree that

$$5 - 2,$$

which names the same number that

$$7$$

does, also names the same number that

$$7$$

does.

## NORA'S SECRETS

If your students have previously studied *Discovery*, which deals with this topic, you may prefer to omit this chapter (or you may prefer to use it for review).

At the start of this lesson, the children presumably have only one method for finding the truth set for the open sentence

$$(\square \times \square) - (5 \times \square) + 6 = 0;$$

this is the method of "trial and error" (or, as the physicist Jerrold Zacharias more suitably describes it, "the method of exploiting errors").

By using this method, they will find that:

"1 does not work":

$$\begin{aligned} & (\boxed{1} \times \boxed{1}) - (5 \times \boxed{1}) + 6 = 0 \\ & \quad \underbrace{1 \quad - \quad 5}_{-4} + 6 = 0 \\ & \quad -4 \quad + \quad 6 = 0 \quad \text{False} \end{aligned}$$

"2 works":

$$\begin{aligned} & (\boxed{2} \times \boxed{2}) - (5 \times \boxed{2}) + 6 = 0 \\ & \quad \underbrace{4 \quad - \quad 10}_{-6} + 6 = 0 \\ & \quad -6 \quad + \quad 6 = 0 \quad \text{True} \end{aligned}$$

"3 works":

$$\begin{aligned} & (\boxed{3} \times \boxed{3}) - (5 \times \boxed{3}) + 6 = 0 \\ & \quad \underbrace{9 \quad - \quad 15}_{-6} + 6 = 0 \\ & \quad -6 \quad + \quad 6 = 0 \quad \text{True} \end{aligned}$$

"4 does not work":

$$\begin{aligned} & (\boxed{4} \times \boxed{4}) - (5 \times \boxed{4}) + 6 = 0 \\ & \quad \underbrace{16 \quad - \quad 20}_{-4} + 6 = 0 \\ & \quad -4 \quad + \quad 6 = 0 \quad \text{False} \end{aligned}$$

One could go on, to find that 5, 6, 7, and 8 do not work, and so on. Notice, however, that this method *by itself* could never fully determine the truth set: we know that 2 and 3 belong to the truth set, *but we do not know that these are the only elements of the truth set*. There are infinitely many numbers, and we can never try them all. Hence, there may always be other elements of the truth set that we have not yet tried.

In the present instance we resolve this difficulty by *telling* the students that there are only two elements in the truth set. (If you would prefer to leave this open-ended, and to avoid the "authoritarian" note of "telling," you can do so.) Hence, once a child has found two numbers that work (i.e., that result in true statements), he knows he has found the entire truth set for that equation.

Now – and this we would *not* tell the children! – if a child is alert and seeking to *discover* patterns, there are two important ones here, just waiting to be discovered:

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$$T = \{2, 3\}$$

Can you find them?



## CHAPTER 10

## Nora's Secrets

[page 35]

Can you find the truth set for these open sentences?

(1)  $\square - 6 = 2$

{ }

(2)  $(2 \times \square) - 6 = 2$

{ }

(3) Can you find the truth set for this open sentence?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

{ . }

Can you find the truth set for each open sentence?

(4)  $(\square \times \square) - (12 \times \square) + 35 = 0$

{ . }

(5)  $(\square \times \square) - (8 \times \square) + 15 = 0$

{ . }

(6)  $(\square \times \square) - (7 \times \square) + 10 = 0$

{ . }

(7)  $(\square \times \square) - (6 \times \square) + 5 = 0$

{ . }

(8)  $(\square \times \square) - (16 \times \square) + 55 = 0$

{ . }

(9)  $(\square \times \square) - (9 \times \square) + 14 = 0$

{ . }

(10) Nora says she knows two secrets about this kind of equation. Do you know what she means?

Can you find the truth set for each open sentence?

(11)  $(\square \times \square) - (15 \times \square) + 26 = 0$

(1) {8}

(2) {4}

(3) {2, 3}

(4) {7, 5}

(5) {3, 5}

(6) {5, 2}

(7) {1, 5}

(8) {11, 5}

(9) {7, 2}

(10) **This is a situation that sorely tries an author's mettle: many people claim that a teacher never feels comfortable letting children learn by discovery unless the teacher himself has experienced what it means to make mathematical discoveries. Perhaps these people are right; on the chance that they may be, we have decided *not* to put this answer in this book.**

(11) {13, 2}

(12)  $(\square \times \square) - (14 \times \square) + 33 = 0$

(12)  $\{11, 3\}$

(13)  $(\square \times \square) - (9 \times \square) + 20 = 0$

(13)  $\{4, 5\}$

Note that 2 and 10 do *not* work; if you don't believe it, substitute into the equation; the resulting statements will be false.

(14)  $(\square \times \square) - (12 \times \square) + 20 = 0$

(14)  $\{2, 10\}$

Here, 4 and 5 do not work.

(15)  $(\square \times \square) - (21 \times \square) + 20 = 0$

(15)  $\{20, 1\}$

(16)  $(\square \times \square) - (15 \times \square) + 36 = 0$

(16)  $\{12, 3\}$

(17)  $(\square \times \square) - (102 \times \square) + 200 = 0$

(17)  $\{100, 2\}$

[page 36]

(18) Do you know Nora's secrets? If you do, DON'T TELL! (It's a SECRET!)

(18) Compare question 10.

Can you find the truth set for each open sentence?

(19)  $(\square \times \square) - (3 \times \square) + 10 = 0$

(19)  $\{5, 2\}$

(20)  $(\square \times \square) - (5 \times \square) + 14 = 0$

(20)  $\{7, 2\}$

(21)  $(\square \times \square) - (5 \times \square) + 6 = 0$

(21)  $\{2, 3\}$

Remember:  $2 \times 3 = 6$ .

(22)  $(\square \times \square) - (9 \times \square) + 22 = 0$

(22)  $\{-11, 2\}$

(23)  $(\square \times \square) - (3 \times \square) + 10 = 0$

(23)  $\{-5, 2\}$

(24)  $(\square \times \square) - (5 \times \square) + 8 = 2$

(24) Ah! This is different. We must first subtract 2 from each side of the equation, to get

$$(\square \times \square) - (5 \times \square) + 6 = 0,$$

for which (as we already know) the truth set is

$$\{2, 3\}.$$

$$(25) \quad (\square \times \square) - (12 \times \square) + 25 \\ = (1 \times \square) + 3$$

(25) Change to

$$(\square \times \square) - (13 \times \square) + 22 = 0,$$

for which the truth set is

$$\{11, 2\}.$$



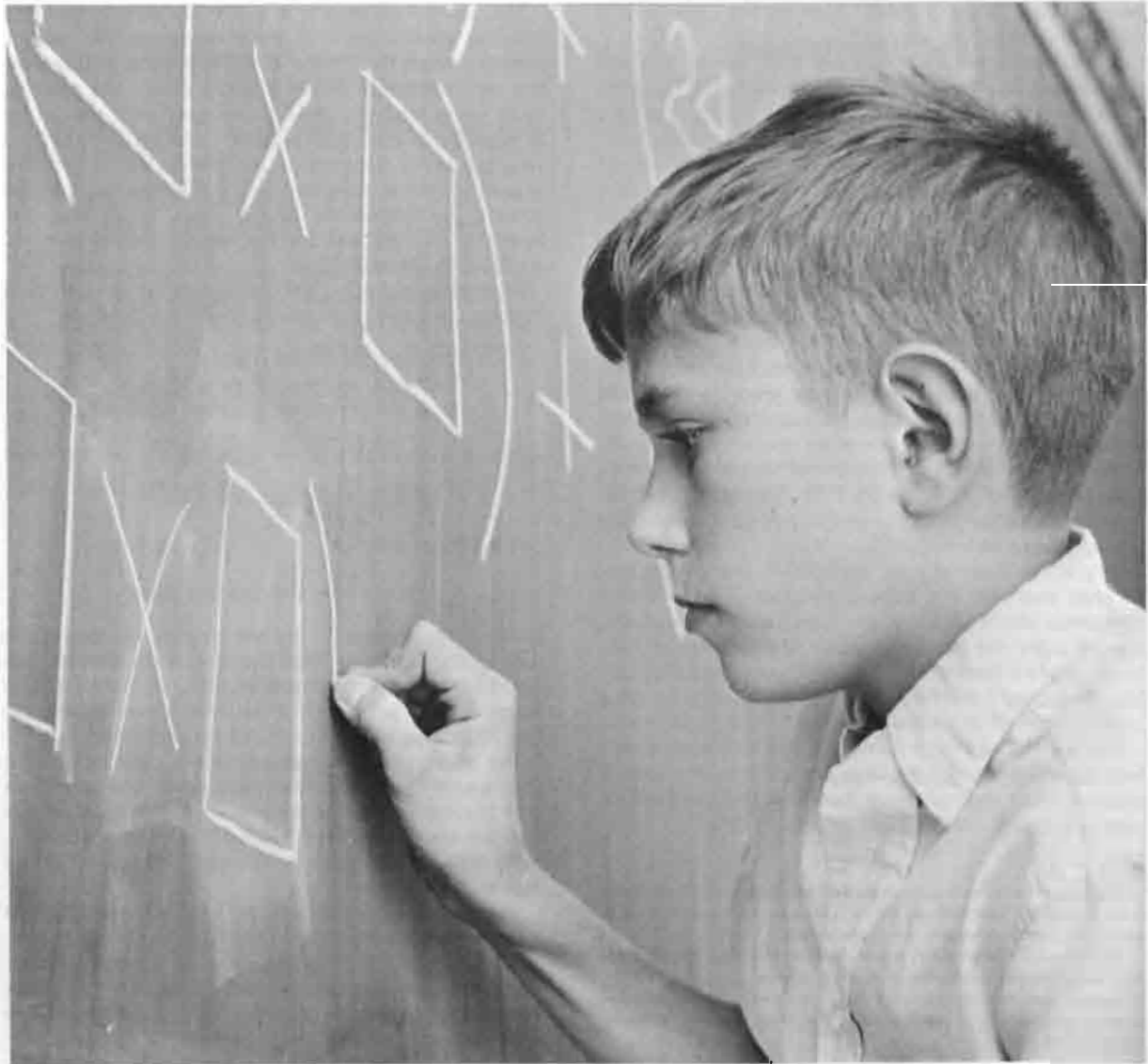
(26)  $(\square \times \square) - (8 \times \square) + 20 = 8$

(26) Change to

$$(\square \times \square) - (8 \times \square) + 12 = 0,$$

for which the truth set is

$\{6, 2\}$ .



## Part Two ■ Logic

We now begin our study of logic. We shall proceed on the assumption that the study of logic is entirely new to you and to your students. The procedure we shall use is more or less typical of how modern scientists construct abstract "cognitive structures," or "models," that enable them to think about complicated aspects of reality.\*

We shall begin by observing how actual people use the words *and*, *or*, *if*, *then*, *not*, and so on. We shall oversimplify this usage drastically. First, we shall focus on the apparent *truth values* that people seem to attach to the various statements. If we let each student make this first "model" by himself, and in his own way—which is what we do in our own Madison Project classes—then we shall get quite a variety of possible models.

Some models will allow only two truth values, *true* and *false* (which we shall often abbreviate as T and F). Some models will allow more truth values, including, perhaps, some like these: "maybe," "not proven," "possible," and so forth. After we have made and looked at these various (somewhat primitive) models, we can work together as a class to build a common model—hoping ultimately to parallel reasonably closely the actual model used most commonly by modern mathematical logicians. This model is a two-valued logic which we shall study in the following pages.

Now, once the bare outline of this model begins to take shape, we can turn to a new kind of question: *we can ask questions about the further development of the model itself, without regard to reality.* (As Professor Marshall Stone has remarked, "Mathematics frees us from the constraints of reality.")

After we have developed the model further, we can turn around and attempt to reapply it to observation of actual behavior of actual people—particularly people who are talking about mathematics.

Why is the study of logic worthwhile? The models that we shall deal with are, indeed, oversimplified versions of the human use of language, so "logic" will not completely and fully *describe* human language. It will, however, help to illuminate a few matters that sometimes need illumination. Moreover, it will help to focus our attention on some of the "little" words we use—the "if's," "or's," and "not's"—that sometimes slip by unheeded, like Volkswagens on a highway much-traveled by large trucks. Even the idea of *truth value* of a statement may help to remind us of one aspect of language that we often forget.

\*Compare the following amusing and perceptive bit of dialogue, from a play by Saul Bellow. Marcella is a woman vacationing in Miami, Ithimar is a leading atomic scientist.

Marcella: . . . You wouldn't know about such things. You take the big view. Overall. International. Cosmic. Nature.

Ithimar: Nature. I don't know nature. I only know certain mathematical structures. . . (Saul Bellow, *The Wen*, *Esquire*, January, 1965, p. 73).

It might be well to look at a few examples. One sometimes hears this usage:

Keep driving like that, and you'll kill somebody.

Probably, by the time we are through building our abstract model, we will interpret the statement above to mean about the same thing as

If you keep driving like that, then you'll kill somebody.

Indeed, our truth-value analysis will probably show these two statements to be *exactly* the same.

Yet, *we can go beyond a truth-value analysis*, and can consider other aspects of these sentences. We can, for example, count the number of words in each. In this case, the two statements do *not* look the same—one contains more words than the other. More importantly, we can try to judge the *tone* of each sentence. Opinions will inevitably vary, but in my own opinion the first statement probably sounds somewhat more forceful, to most hearers, than the second. The moment we use the word *if*, we invite our listeners to sit back and disassociate themselves from our words—"Ah, he said *if* . . ."

As another example, consider the public-service notice that was used over radio stations prior to the 1964 elections:

Vote, and the choice is yours.  
Don't vote, and the choice is theirs.  
If you fail to register, you have no choice.

Again, a truth-value analysis might regard the statement

Vote, and the choice is yours

and the statement

If you vote, the choice is yours

as identical. However, in my own opinion, the tone of the first is much stronger than the tone of the second. One might say that the nuances of meaning are different. This example, then, suggests that the usual uses of logic do *not* fully reflect the entire meaning of our statements.

The actual use of human language is complex, indeed. Even a vastly simplified description, which is what we here seek to construct, can be of some value—much as a hastily drawn map can sometimes be. The "logic" we are developing is really, then, a rough description of one aspect of how people use language.

Why is this worth studying? My own answer is based on observing secretaries, delivery men, and all kinds of people, in a wide variety of situations. Observation makes it clear that people use language carelessly, and especially where words like *or* or *not* are concerned. If you conduct your own observations, I think you can find plenty of examples.

Here are a few instances where the usual use of such words is careless, or where such carelessness has led to trouble:

(i) A letter said: "If you have a contract with Mr. W\_\_\_\_\_, we would like to examine a copy." This letter was construed to mean that the recipient *should have a* contract with Mr. W\_\_\_\_\_. Perhaps it did mean this, but it didn't say so.

(ii) Professor Layman Allan, of the Yale Law School, has pointed out that *logical ambiguities* abound in supposedly careful legal documents. It is not hard to find some in documents released by state governments, in contracts of various sorts, and so on.

(iii) A world-famous mathematician went to take a driving examination. He had memorized statements from a booklet, including the statement

It is illegal to park within  
15 feet of a fire hydrant.

As part of the test, he was given some "true-false" questions, including this one:

It is illegal to park within 9 feet of a fire hydrant.	<input type="checkbox"/> True <input type="checkbox"/> False
---	---

The mathematician checked "true," on the grounds that if the first statement was true, the second surely was. The examiner, however, claimed that the correct choice was "false," since "it should be fifteen feet, not nine feet."

(iv) If you whisper a statement such as

The moon is high, or John is not ready

to a student, ask him to whisper it to another student, and

ask the second student to repeat the statement aloud, you will find errors often creep in. A particularly common error is to change the "or" to an "and," and to end up with the statement

The moon is high, and John is not ready.

This, however, involves a major alteration in the logical meaning of the statement.

(v) In the "game of clues," which we shall encounter in Chapter 16, the following situation sometimes arises. The children are told the "clue,"

All of the numbers are odd.

They subsequently learn that this clue was false. Consequently, they change it to read

All of the numbers are even.

This, however, is not a correct "negation" or "denial" of the original clue. What they should write is

At least one of the numbers is even.

It seems reasonable to hope that a brief study of logic may help to improve the use of these much-neglected (but important) "little" words, such as *if, not, and, or,* and so on.\*

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\*Some results of Professor Bruner's Center for Cognitive Studies at Harvard give reason for optimism. Merely considering alternatives can improve one's response to a situation—the most limiting "mental set" apparently often results from never having considered alternatives in the first place. See Appendix A: Weil (143).

## LOGIC (BY OBSERVING HOW PEOPLE USE WORDS)

Notice carefully the approach to logic which is used in this chapter. We begin by looking at *people*—specifically, the people around us—and observing how *they* use the words *and*, *or*, *not*, *if*, *then*, etc. We are not concerned with how they *claim* they use these words, but rather with how they *actually do use these words*. In this sense, one might say that this chapter is concerned with *sociology* or with *cultural anthropology*: we are occupied with the task of “observing the natives.”

If we observe the natives honestly and carefully, we shall find that:

(i) They use the words *and*, *or*, etc., in a variety of different ways; they are *not* always consistent. (Indeed, it might not be too extreme to say that they are downright “sloppy” and inexact.)

(ii) In most cases, we take our meanings *more from the context* than from the explicit statement.

(iii) Any attempt to develop a simple description of the use of language must, in fact, be an *oversimplification* of the actual complicated reality.

In the next few chapters, our program for the development of *logic* will go through three stages. First, we shall study how people actually use the words *and*, *or*, *true*, *false*, *if*, *then*, etc. We shall represent this usage by an oversimplified model, using a two-valued logic. Second, we shall give greater precision to *our* use of these words, by making some agreements on how we shall use these words henceforth in this course. Finally, we shall take a “mathematician’s eye view” of what is going on. We shall make everything abstract, remove virtually all vestiges of context, and ask how we can study and extend the abstract mathematical system with which we seem to be dealing.

In what respects is our description in this chapter oversimplified? Perhaps primarily in these two regards:

*Linguistic Simplification.* The placement of *not* within an English statement is a subtle and tricky matter. Consider, for example, the statement

... where the skies are not cloudy all day.

- Does this mean that, all day long, there is never a cloud in the skies? Does it mean that the skies aren’t cloudy *all* day: there is a brief period around 2 o’clock in the afternoon when a bit of blue sky and sun breaks through? Does it mean that, generally speaking, the skies are *less* cloudy than they are back East? What, in fact, does it mean?

Within our representation of statements, we shall take a simple statement

$$P$$

and denote its negation by

$$\sim P,$$

which we can read as "not  $P$ ." We clearly oversimplify English a great deal when we treat negation in such a simple fashion.

*Truth-Value Simplification.* Most statements in ordinary conversation are really not "absolutely true," nor are they "absolutely false." They usually contain "some truth," but also admit some room for disagreement. Consider, for example, the statement "red is a pretty color," or the statement "the United States is a relatively young nation." By the same token, the negation of most statements is not "absolutely true," nor "absolutely false." There are, in ordinary conversation, many shades of "truth" and "falsity." Yet, in order to get a simple model, we shall regard all of our statements as either true or false. This is known as imposing upon our statements a "two-valued logic": the only allowable values will be "true" and "false." Clearly this is an oversimplification of the actual reality.

For further reading, see Appendix A: Allendoerfer (76), Davis (27), Exner (40), Kemeny (82), Ohmer (97), Mendelson (93), and Eves (86).



## CHAPTER 11

## Logic (by observing how people use words)

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(1) Earle wrote

$$\square + \triangle = 8;$$

$$R_{\square} = \{2, 3, 5, 7\};$$

$$R_{\triangle} = \{1, 3, 5\}.$$

$\square$	$\triangle$
3	5
5	3
2	5
7	1
1	7

Table for  
Truth Set

What do you think?

## ANSWERS AND COMMENTS

Notice that we use subscripts to indicate which replacement set is to correspond to which variable.

(1) **If we consider only Earle's replacement sets and do not ask ourselves which statements are true and which are false, we see that the statements**

$$\square + \triangle = 8$$

$$R_{\square} = \{2, 3, 5, 7\}$$

$$R_{\triangle} = \{1, 3, 5\}$$

mean exactly the same thing as

$$2 + 1 = 8$$

$$3 + 1 = 8$$

$$5 + 1 = 8$$

$$7 + 1 = 8$$

$$2 + 3 = 8$$

$$3 + 3 = 8$$

$$5 + 3 = 8$$

$$7 + 3 = 8$$

$$2 + 5 = 8$$

$$3 + 5 = 8$$

$$5 + 5 = 8$$

$$7 + 5 = 8$$

So far we have used only the ideas of *variables* and *replacement sets*. However, we are interested in the *truth set*.

Of the 12 preceding statements the following are true:

$$\begin{aligned} 7 + 1 &= 8 \\ 5 + 3 &= 8 \\ 3 + 5 &= 8 \end{aligned}$$

The remaining 9 statements are false. Consequently, the *truth set* is given by the following table:

$\square$	$\triangle$
7	1
5	3
3	5

Earle's table for the truth set is wrong. It contains two listings which should not be there. The replacement

$$2 \rightarrow \square \text{ and } 5 \rightarrow \triangle$$

is incorrect because

$$2 + 5 \neq 8.$$

The replacement

$$1 \rightarrow \square \text{ and } 7 \rightarrow \triangle$$

is incorrect, even though  $1 + 7 = 8$ , because 1 is not an element of the replacement set of  $\square$ .

(2) Joan wrote

$P$  is  $x$ ;

$R_p = \{\text{"New Orleans is a city"}, \text{"New Hampshire is a city"}, \text{"New Jersey is a state"}\};$

$R_x = \{\text{true}, \text{false}\}.$

Can you make a table to show the truth set for Joan's open sentence?

(2) In the first place, thinking only of Joan's *replacement sets* and *not* asking ourselves, just yet, which statements are "true" and which are "false," we see that the statements

$P$  is  $x$ ;

$R_p = \{\text{"New Orleans is a city"}, \text{"New Hampshire is a city"}, \text{"New Jersey is a state"}\};$

$R_x = \{\text{true}, \text{false}\}$

mean exactly the same thing as:

"New Orleans is a city" is true.

"New Orleans is a city" is false.

"New Hampshire is a city" is true.

"New Hampshire is a city" is false.

"New Jersey is a state" is true.

"New Jersey is a state" is false.

Thusfar, we have used only the ideas of *variable* and *replacement sets*. But we are asked to think about the *truth set*—i.e., we are asked which of these uses of UV led to *true* statements. Evidently, of the six statements above, the first, fourth, and fifth are *true*:

"New Orleans is a city" is true.

"New Hampshire is a city" is false.

"New Jersey is a state" is true.

On the other hand, we have three false statements:

"New Orleans is a city" is false.  
 "New Hampshire is a city" is true.  
 "New Jersey is a state" is false.

Consequently, the *truth set* is given by the following table:

$P$	$x$
"New Orleans is a city"	True
"New Hampshire is a city"	False
"New Jersey is a state"	True

We could also write this *truth set* as

$$T = \{("New Orleans is a city", \text{true}), ("New Hampshire is a city", \text{false}), ("New Jersey is a state", \text{true})\}.$$

Notice that  $T$  is a set of ordered pairs, and that, in fact,  $T$  is a subset of the Cartesian product

$$R_p \times R_x,$$

which we could write as

$$T \subset R_p \times R_x.$$

This example is a tricky one. If our students are to like mathematics, the "light touch" is important. Do not dwell on this problem. In fact, if you prefer, leave it out altogether!

Indeed, you may prefer to make up an easier example of your own to put in its place. But, whatever you do, *please don't try to "drive this example home."* You can only alienate children by such an approach.

(3) Larry made this table for the truth set of Joan's open sentence.

$P$	$x$
New Orleans is a city.	true
New Hampshire is a city.	false
New Hampshire is a city.	true
New Jersey is a state.	false

Do you agree?

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(4) John said he was going to keep track of his friends' statements. He was **not** going to see how many words they used, he was **not** going to worry whether their words were elegant, and he was **not** going to care about what they said. The **only thing** he was going to study was whether their statements were true or false. John made this table:

$\square$	$\triangle$	$\square$ and $\triangle$

He said, "I'm going to study what my friends mean when they use the word 'and.' I don't know what statement they may put in the  $\square$ , but it will be either true

(3) **Larry's table is wrong. A correct table is shown above, in the answer to question 2.**

(4) **Four. The statement in the  $\square$  may be either true or false, and the statement in the  $\triangle$  may be either true or false. Hence, there are these four possibilities:**

$\square$	$\triangle$
T	T
T	F
F	T
F	F

or false. I don't know what statement they may put in the  $\triangle$ , but it will be either true or false."

How many possibilities must John allow for in his table?

(5) Eileen said that people might put a true statement in the  $\square$  and a true statement in the  $\triangle$ , so she wrote:

$\square$	$\triangle$	$\square$ and $\triangle$
T	T	

What do you think?

(6) Jill said that people might put a true statement in the  $\square$  and a false statement in the  $\triangle$ , so she added another line to John's table:

$\square$	$\triangle$	$\square$ and $\triangle$
T	T	
T	F	

What do you think?

(7) Can you add any more possibilities to John's table?

(8) John asked one of his friends to make up a sentence,

$\square$  and  $\triangle$ ,

with

$R_1 = \{ \text{"I am ten years old", "I am fifteen years old", "I am seven feet tall"} \}$

$R_2 = \{ \text{"my name is George", "my name is Albert"} \}$ .

How many sentences could his friend have written? Can you write them all?

(9) Henry, who is twelve years old, wrote:

I am ten years old and my name is Albert.

Was Henry's statement true or false?

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(10) Nancy says that if you put a true statement in the  $\square$  and a true statement in the  $\triangle$ , then the statement

$\square$  and  $\triangle$

will be true; so she wrote:

$\square$	$\triangle$	$\square$ and $\triangle$
T	T	T

What do you think?

(5) Eileen has one of the four possibilities.

(6) Jill is correct.

(7) Compare the answer to question 4.

(8) With  $R_1$  and  $R_2$  as given in the question,

$\square$  and  $\triangle$

represents these statements:

- I am ten years old and my name is George.
- I am ten years old and my name is Albert.
- I am fifteen years old and my name is George.
- I am fifteen years old and my name is Albert.
- I am seven feet tall and my name is George.
- I am seven feet tall and my name is Albert.

There are six statements, as there should be, since we are dealing with the Cartesian product  $R_1 \times R_2$ , where there are three elements in  $R_1$  and two elements in  $R_2$ .

(9) For him, the statement was false.

(10) Nancy is evidently using the word *and* in its most common sense. She is certainly correct, according to the way most people most often use the word *and*.



(11) Can you complete John's table?

(11) There are many possible ways to complete John's table, because there are many different uses of the word *and*. Probably the most common would be:

□	△	□ and △
T	T	T
T	F	F
F	T	F
F	F	F

If, for example, someone promised us "\$5 and a year's subscription to the *St. Louis Post-Dispatch*" we would feel he had kept his promise only if we received *both* the \$5 *and* the subscription to the *Post-Dispatch*.

Other uses of the word *and* do, however, exist. An ingenious teacher suggested the following:

"You keep driving like that and you'll kill somebody."

Now, when would we consider this statement true? If we *do* keep driving like that, and we *do* kill somebody, we'd probably ruefully admit the prediction was correct, so we have

□	△	□ and <sub>2</sub> △
T	T	T

where we have used "*and*<sub>2</sub>" to emphasize that this is a *second* meaning for *and*, quite different from the earlier usage.

Suppose we *do* keep driving like that, and we *don't* kill anybody. Then we might argue that our critic was wrong; hence we might write:

□	△	□ and <sub>2</sub> △
T	T	T
T	F	F

Suppose we *stop* driving "like that," and we kill somebody anyhow. My guess is that most people would then argue that the advice had been wrong:

□	△	□ and <sub>2</sub> △
T	T	T
T	F	F
F	T	F

Finally, suppose we *stop* driving "like that" and we do not kill anybody. We might then feel that the advice had come "just in time," and might represent the abstract analysis of this case as:

□	△	□ and <sub>2</sub> △
T	T	T
T	F	F
F	T	F
F	F	T

There are, in fact, still other uses of the word *and* that you can locate if you seek them diligently. Can you identify an "*and*<sub>3</sub>"?

(12) Can you make a table of the way your friends use the word "or"?

(12) There are two uses of *or* that are about equally common. When someone says

"I'd love to get an A in math or English,"

I think he means he would love an A in math, he would love an A in English, and he would love an A in both subjects. The truth table for this use of *or*—which we shall call "*or*<sub>1</sub>"—goes like this:

□	△	□ or <sub>1</sub> △
T	T	T
T	F	T
F	T	T
F	F	F

There is a second use of *or*, which we shall call "*or*<sub>2</sub>." When a restaurant menu says

one vegetable or salad,

I think it means that you may have one vegetable or you may have salad but you don't get both. The truth table for "*or*<sub>2</sub>" would then be:

□	△	□ or <sub>2</sub> △
T	T	F
T	F	T
F	T	T
F	F	F

Can you find any other uses of the word *or*?

(13) Sandy made this table to show how her friends use "*or*":

<i>P</i>	<i>Q</i>	<i>P</i> or <i>Q</i>
T	T	T
T	F	T
F	T	T
F	F	F

What do you think?

(13) Sandy's table is correct; this use of *or* is often called the *inclusive or*. It is this use of *or* which is always used in mathematics.

(14) Ann disagreed with Sandy. Ann says her friends use "*or*" this way:

<i>P</i>	<i>Q</i>	<i>P</i> or <i>Q</i>
T	T	F
T	F	T
F	T	T
F	F	F

What do you think?

(14) Ann's table is also correct; this use of *or* is often called the *exclusive or*. Apparently this use of *or* is implicit among lawyers, or else they would have no need for the symbol *and/or* which is common in legal usage.

(15) Alex says sometimes his friends use "or" the way Sandy says and sometimes the way Ann says. What do you think?

(16) Alex gave this example: "I'll either go canoeing all day Saturday or I'll go to the baseball game." Which way is "or" used in this sentence, Ann's way or Sandy's way?

(17) Kevin gave this example: "I sure hope I get an A in English or math." Which way is "or" used in this sentence?

(18) A waitress said, "You may have potato or spaghetti." Which kind of "or" did she mean?

(19) Do you know what mathematicians mean by the "inclusive or"?

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(20) Do you know what mathematicians mean by the "exclusive or"?

(21) Kathy made a table for her symbol " $\sim$ ," which means "not":

$P$	$Q$	$\sim P$
T	T	F
T	F	F
F	T	T
F	F	T

Do you agree?

(22) Can you make a table for " $\sim Q$ "?

(15) Alex is, of course, correct.

(16) This is presumably the exclusive or (i.e., Ann's version), since Alex presumably means he will *not* do both.

(17) This is presumably the inclusive or (i.e., Sandy's version).

(18) Presumably the exclusive or (i.e., you don't get both potato and spaghetti).

(19) and (20) Compare the preceding questions.

(21) Kathy is correct.

(22)

$P$	$Q$	$\sim Q$
T	T	F
T	F	T
F	T	F
F	F	T

(23) John uses  $\square$  and  $\triangle$  for the variables in his tables. Sandy uses  $P$  and  $Q$  for the variables in her table. In order not to get mixed up, John and Sandy have made a table labeled with both  $P$  and  $Q$  and  $\square$  and  $\triangle$ :

		"Inclusive or"		"Exclusive or"		
$\square$	$\triangle$	$\square$ and $\triangle$	$\square$ or $\triangle$	$\square$ or $\triangle$	$\sim \square$	$\sim \triangle$
$P$	$Q$	$P$ and $Q$	$P$ or $Q$	$P$ or $Q$	$\sim P$	$\sim Q$
T	T					
T	F					
F	T					
F	F					

Can you fill in the rest of their table?

(23)

		"Inclusive or"		"Exclusive or"		$\sim \square$	$\sim \triangle$
$\square$	$\triangle$	$\square$ and $\triangle$	$\square$ or $\triangle$	$\square$ or $\triangle$	$\sim \square$	$\sim \triangle$	
$P$	$Q$	$P$ and $Q$	$P$ or $Q$	$P$ or $Q$	$\sim P$	$\sim Q$	
T	T	T	T	F	F	F	
T	F	F	T	T	F	T	
F	T	F	T	T	T	F	
F	F	F	F	F	T	T	

(24) What do your friends mean by 'if ... then ...'? Can you show this by a truth table?

$\square$	$\triangle$	If $\square$ , then $\triangle$ .
$P$	$Q$	If $P$ , then $Q$ .
T	T	
T	F	
F	T	
F	F	

(Mathematicians write "If  $P$ , then  $Q$ " this way:

$$P \Rightarrow Q$$

or else

$$P \supset Q.)$$

(24) There is considerable variation among uses of *if ... then*. Consequently you should expect a number of different analyses from your students. Here are some common ones:

$\square$	$\triangle$	If $\square$ , then, $\triangle$ .
$P$	$Q$	If $P$ , then, $Q$ .
T	T	T
T	F	F
F	T	F
F	F	?

We might, for example, be thinking of the statement "If I wash my car, then it will rain." In the fourth case, FF, we might feel that we had never properly put the matter to a test. Notice, however, that if we accept a table such as this, we have abandoned our two-valued logic, and are now allowing three symbols: T, F, and ?.

The following table is a version of our first table, modified to reject the "?" in order to retain a two-valued logic.

$P$	$Q$	If $P$ , then, $Q$ .
T	T	T
T	F	F
F	T	F
F	F	T

Evidently, we might instead have chosen to modify table 1 as follows:

$P$	$Q$	If $P$ , then, $Q$ .
T	T	T
T	F	F
F	T	F
F	F	F

You might say that when we never did wash our car, and so never did test out the proposition, table 2 gives it "the benefit of the doubt," whereas table 3 rejects it as "not proved."

Mathematicians *always* use *if ... then* according to the following table:

$P$	$Q$	If $P$ , then, $Q$ .
T	T	T
T	F	F
F	T	T
F	F	T

This meaning appears also in legal contexts; if my insurance policy says

"If I die, the insurance company will pay my estate \$30,000," then if I do die and they do pay, they have not violated the contract; if I don't die and they do pay, they have not violated our contract

(although company stockholders may feel that the management has been overgenerous); if I do die and they don't pay, they *have* violated the contract (my trustees go to court!), and if I don't die and they don't pay, they have not violated the contract.

(25) Sandy's father says that mathematicians write

$$P \Leftrightarrow Q$$

to mean

$P$  has the same truth value as  $Q$ .

Sandy made a truth table for

$$P \Leftrightarrow Q.$$

Can you figure out how she did it?

$P$	$Q$	$P \Leftrightarrow Q$
T	T	
T	F	
F	T	
F	F	

(25)

$P$	$Q$	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that, in this chapter, we don't want the students trying to get the "right" answer. We want them to seek an honest analysis of how their friends use these various words. Consequently, they may come up with answers quite different from those suggested here.

## LOGIC (BY MAKING AGREEMENTS)

In the preceding chapter we outlined our approach to logic, which might be described as a three-step approach:

- Step 1: Cultural anthropology
- Step 2: Legislation
- Step 3: Mathematical exploration

Chapter 11 was concerned with step 1, the cultural anthropology or linguistics approach; we tried to see how people actually do seem to use the words *and*, *or*, etc; and *what they seem to mean*. At this stage the goal is honest, shrewd *observation*. We want to observe *actual usage* of these words by *actual people*. We do not want—heaven forbid!—to color our observations with *our* expectations of how people “ought to use these words and what they ought to mean.” As a result, there is no “right” answer in Chapter 11 that can be predicted in advance. The goal is for each child to be true to his own experience. We have started, already in Chapter 11, to make a very simple model for this linguistic behavior, in terms of two-valued logic and truth tables.

Now, in the present chapter, we begin step 2: legislation by making agreements. We shall take the diverse and chaotic usages of Chapter 11 and impose order and clarity by legislative edict. We shall agree that, henceforth in this course, we shall use the various words in strict accordance with a single meaning, as revealed in a single column of the truth table. By this method we shall achieve the incredible result—as Professor Patrick Suppes points out—of *using an imprecise language to create a more precise language!*



### CHAPTER 12

## Logic (by making agreements)

[page 41]

(1) Bill says it is very confusing to find people using the same word for different meanings. He says we should agree, in this class, that whenever we say “and” we will use it according to this table:

<i>P</i>	<i>Q</i>	<i>P</i> and <i>Q</i>
T	T	T
T	F	F
F	T	F
F	F	F

What do you think?

## ANSWERS AND COMMENTS

- (1) **This is a good idea. It initiates step 2 of our three-step approach to logic.**

(2) Paul's father says that mathematicians always use "or" to mean the "inclusive or." Let's complete the following truth table for "or." And let's agree that, in this class, we will always use "or" according to our table.

$P$	$Q$	$P$ or $Q$

(3) Let's make up a truth table for "if ... then ...". Let's agree that, in this class, we will always use "if ... then ..." according to the table we make up.

$P$	$Q$	If $P$ , then $Q$ .

(4) Let's make up truth tables for " $\sim$ ," and for " $\Leftrightarrow$ ," and let's agree that, from now on in this class, we will use " $\sim$ " and " $\Leftrightarrow$ " according to our tables.

$P$	$Q$	$\sim P$	$\sim Q$	$P \Leftrightarrow Q$

(2) It is important in *this chapter* to make sure that our agreements are consistent with those commonly made by mathematicians and logicians. Here is the truth table:

$P$	$Q$	$P$ or $Q$
T	T	T
T	F	T
F	T	T
F	F	F

(3) Here is the table which mathematicians use:

$P$	$Q$	If $P$ , then $Q$ .
T	T	T
T	F	F
F	T	T
F	F	T

This table may seem "peculiar"—perhaps even *wrong*—to you and to your students. This is presumably due to the fact that *if ... then* is used in many different ways in ordinary usage, and in yet one more way by mathematicians. Notice especially that none of the usual notions of "cause" and "effect" are included in the mathematical usage.

This is deliberate; as long as *causality* is involved, there can never be certainty as to whether a statement is true or false. (For example, does some kind of virus cause cancer? Does motion through the ether cause a shortening of measuring rods, as people believed before Einstein? Would driving 30 miles per hour cause you to drop dead, as many people predicted when trains and automobiles were first introduced?) We need a more abstract theory, one that is *not too close to reality*. We do not know reality, but we can make up abstract systems ourselves, and we can be reasonably sure about them.

The symbols for "If  $P$ , then  $Q$ " are either

$$P \Rightarrow Q$$

or

$$P \supset Q.$$

(4) Here is the truth table, as used by mathematicians and logicians:

$P$	$Q$	$\sim P$	$\sim Q$	$P \Leftrightarrow Q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

## SOME COMPLICATED FORMULAS IN LOGIC

Because this chapter is possibly somewhat more difficult than any thusfar, we recommend that, when in doubt, you omit it.

If you do include this chapter, please try to go through it "lightly." This is a real test of the "light touch." A heavy-handed, "systematic," pedantic treatment of this chapter will almost certainly not work with younger children. A light, "intuitive" approach, however, will work.

Mathematically, there are two points to this chapter:

(i) We can use our existing knowledge to fill in many additional columns in our truth table—columns with headings such as:

$$\sim (P \text{ and } Q), \quad \sim (P \text{ or } Q), \quad (\sim P) \text{ or } (\sim Q), \quad (\sim P) \text{ or } Q.$$

(ii) When we do so, *some columns will be the same*. We shall not say very much about this, leaving it instead for student discovery.

Before teaching this lesson, you may want to view the Madison Project film entitled "Extending Truth Tables."



### CHAPTER 13

## Some Complicated Formulas in Logic

[page 42]

- (1) Larry says he can fill in the column for  $\sim (P \text{ and } Q)$ .

Can you?

(2) Joan says there is really nothing new in this; you just use things you already know, such as " $\sim$ " and "and." What do you think?

- (3) Can you fill in a column for  $\sim (P \text{ or } Q)$ ?

## ANSWERS AND COMMENTS

- (1) **Actually, this is now fully determined by what has gone before. Here is the result:**

P	Q	P and Q	$\sim (P \text{ and } Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

- (2) **Joan is correct.**

- (3)

P	Q	P or Q	$\sim (P \text{ or } Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T



- (4) Can you fill in a column for  $\sim(\sim P)$ ?

(4)

P	Q	$\sim P$	$\sim(\sim P)$
T	T	F	T
T	F	F	T
F	T	T	F
F	F	T	F

At this point some students may already have made the important discovery that the columns for  $P$  and for  $\sim(\sim P)$  are *exactly the same*. If any students *do* discover this, you may want to encourage them to try writing their discovery. They can do this as follows:

$$P \Leftrightarrow [\sim(\sim P)].$$

(Remember that " $\Leftrightarrow$ " means "has the same truth value as.")

- (5) Can you fill in a column for  $\sim(P \Rightarrow Q)$ ?

(5)

P	Q	$P \Rightarrow Q$	$\sim(P \Rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

- (6) Can you fill in a column for  $\sim(P \Leftrightarrow Q)$ ?

(6)

P	Q	$P \Leftrightarrow Q$	$\sim(P \Leftrightarrow Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

- (7) Can you fill in a column for  $(\sim P) \text{ and } (\sim Q)$ ?

- (7) We shall do this in several stages. First,

$\square$  and  $\triangle$

is *true* only if we put a true statement into the  $\square$  and a true statement into the  $\triangle$ . When will we do this? Evidently, if  $\sim P$  is true and if  $\sim Q$  is true:

P	Q	$\sim P$	$\sim Q$	$(\sim P) \text{ and } (\sim Q)$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	T

Now, in all *other* cases, " $\square$  and  $\triangle$ " will be false:

P	Q	$\sim P$	$\sim Q$	$(\sim P) \text{ and } (\sim Q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

- (8) Can you fill in a column for  $(\sim P) \text{ or } (\sim Q)$ ?

Can you fill in columns with the following headings?

- (8) This, too, we shall do in stages. First,

$\square$  or  $\triangle$

becomes false if we put a false statement into  $\square$  and a false statement into  $\triangle$ . When will this happen? If  $\sim P$  is false, and if  $\sim Q$  is false:

P	Q	$\sim P$	$\sim Q$	$(\sim P) \text{ or } (\sim Q)$
T	T	F	F	F
T	F	F	T	
F	T	T	F	
F	F	T	T	

In all other cases, " $(\sim P) \text{ or } (\sim Q)$ " will be true:

P	Q	$\sim P$	$\sim Q$	$(\sim P) \text{ or } (\sim Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

- (9)  $(\sim P) \text{ or } Q$

- (9) Again, we work in stages: " $(\sim P) \text{ or } Q$ " will be false if both  $(\sim P)$  and  $Q$  are both false:

P	Q	$\sim P$	$(\sim P) \text{ or } Q$
T	T	F	
T	F	F	F
F	T	T	
F	F	T	

Otherwise, " $(\sim P) \text{ or } Q$ " will be true:

P	Q	$\sim P$	$(\sim P) \text{ or } Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Here, also, a student may discover that the column for

$(\sim P) \text{ or } Q$

is exactly the same as the column for

$P \Rightarrow Q$ .

If anyone can write this, we usually name it after the student who first writes it correctly; for example,

Toby's formula:  $[(\sim P) \text{ or } Q] \Leftrightarrow [P \Rightarrow Q]$ .

(10)  $P$  or  $(\sim Q)$ (10) Here, also, we shall work in stages. In the first place, " $P$  or  $(\sim Q)$ " is false if  $P$  is false and if  $(\sim Q)$  is false:

$P$	$Q$	$\sim Q$	$P$ or $(\sim Q)$
T	T	F	
T	F	T	
F	T	F	F
F	F	T	

In all other cases, " $P$  or  $(\sim Q)$ " will be true:

$P$	$Q$	$\sim Q$	$P$ or $(\sim Q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(11)  $Q \Rightarrow P$ (11) If you "promise and don't deliver,"  $Q \Rightarrow P$  is false:

$P$	$Q$	$Q \Rightarrow P$
T	T	
T	F	
F	T	F
F	F	

In all other cases, it is true:

$P$	$Q$	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

Suppose Jackie discovers the following formula and writes it correctly; then we have

Jackie's Formula:  $[Q \Rightarrow P] \Leftrightarrow [P \text{ or } (\sim Q)]$ .

(12)  $P$  and  $(\sim Q)$ (12) Again, we work by stages. " $P$  and  $(\sim Q)$ " is true if  $P$  is true and  $(\sim Q)$  is true:

$P$	$Q$	$\sim Q$	$P$ and $(\sim Q)$
T	T	F	
T	F	T	T
F	T	F	
F	F	T	

Otherwise, "P and (~ Q)" is false:

P	Q	~ Q	P and (~ Q)
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(13)  $(\sim P) \Leftrightarrow Q$

(13) We need only fill in columns for (~ P) and for Q, and then see where they *do* have the same truth values [(~ P) ⇔ Q will be true] and where they *don't* [(~ P) ⇔ Q will be false]:

P	Q	~ P	(~ P) ⇔ Q
T	T	F	
T	F	F	T
F	T	T	T
F	F	T	

P	Q	~ P	(~ P) ⇔ Q
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

(14)  $\sim [(\sim P) \Leftrightarrow Q]$

(14) This is an easy one. Just use the column for (~ P) ⇔ Q:

P	Q	(~ P) ⇔ Q	~ [(~ P) ⇔ Q]
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	T

(15)  $\sim \{ \sim [(\sim P) \Leftrightarrow Q] \}$

(15) Another easy one; just use the column for ~ [(~ P) ⇔ Q]:

P	Q	~ [(~ P) ⇔ Q]	~ { ~ [(~ P) ⇔ Q] }
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Notice that we are accumulating more and more opportunities for discoveries. Here are a few:

Jill's formula:  $\{ \sim [(\sim P) \Leftrightarrow Q] \} \Leftrightarrow [P \Leftrightarrow Q]$

John's formula:  $[ \sim \{ \sim [(\sim P) \Leftrightarrow Q] \} ] \Leftrightarrow [(\sim P) \Leftrightarrow Q]$

Bernice's formula:  $\{ \sim [(\sim P) \text{ or } Q] \} \Leftrightarrow [P \text{ and } (\sim Q)]$

Evelyn's formula:  $[(\sim P) \text{ or } Q] \Leftrightarrow \{ \sim [P \text{ and } (\sim Q)] \}$

(16)  $[(\sim P) \Leftrightarrow Q] \text{ or } [P \Rightarrow Q]$

- (16) This, too, is fairly easy. We use the columns for  $(\sim P) \Leftrightarrow Q$  and for  $P \Rightarrow Q$ , and use the fact that " $\square$  or  $\triangle$ " will be false only if we put a false statement into the  $\square$  and a false statement into the  $\triangle$ :

P	Q	$(\sim P) \Leftrightarrow Q$	$P \Rightarrow Q$	$[(\sim P) \Leftrightarrow Q] \text{ or } [P \Rightarrow Q]$
T	T	F	T	
T	F	T	F	
F	T	T	T	
F	F	F	T	

However,  $(\sim P) \Leftrightarrow Q$  and  $P \Rightarrow Q$  are never simultaneously false! Consequently, " $[(\sim P) \Leftrightarrow Q] \text{ or } [P \Rightarrow Q]$ " is never false; it must always be true:

P	Q	$(\sim P) \Leftrightarrow Q$	$P \Rightarrow Q$	$[(\sim P) \Leftrightarrow Q] \text{ or } [P \Rightarrow Q]$
T	T	F	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(17)  $[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]$

- (17) Again, we work by stages. First, we shall need the columns for  $P \Rightarrow Q$  and for  $Q \Rightarrow P$ . Then, we shall recall that " $\square$  or  $\triangle$ " is false only if we put a false statement into the  $\square$  and a false statement into the  $\triangle$ :

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]$
T	T	T	T	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Again, we see that  $P \Rightarrow Q$  and  $Q \Rightarrow P$  are never simultaneously false; hence " $[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]$ " must never be false! It, also, is always true:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(18)  $\sim \{[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]\}$

- (18) We merely use the preceding column, and here we have an expression which is always false:

P	Q	$[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]$	$\sim \{[P \Rightarrow Q] \text{ or } [Q \Rightarrow P]\}$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	F

(19)  $P \Leftrightarrow (\sim Q)$

$P$	$Q$	$\sim Q$	$P \Leftrightarrow (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

We have the possibility of another discovery:

Miriam's formula:  $[(\sim P) \Leftrightarrow Q] \Leftrightarrow [P \Leftrightarrow (\sim Q)]$

(20)  $\sim [P \Leftrightarrow (\sim Q)]$

$P$	$Q$	$P \Leftrightarrow (\sim Q)$	$\sim [P \Leftrightarrow (\sim Q)]$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	F	T

(21)  $P \text{ and } (\sim P)$

$P$	$Q$	$\sim P$	$P \text{ and } (\sim P)$
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	F

Actually, of course,  $Q$  plays no role in this problem, and we could use a shorter truth table:

$P$	$\sim P$	$P \text{ and } (\sim P)$
T	F	F
F	T	F

(22)  $P \text{ or } (\sim P)$

$P$	$Q$	$\sim P$	$P \text{ or } (\sim P)$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	T

Using the shorter table:

$P$	$\sim P$	$P \text{ or } (\sim P)$
T	F	T
F	T	T

(23)  $P \Leftrightarrow (\sim P)$

$P$	$Q$	$\sim P$	$P \Leftrightarrow (\sim P)$
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	F

Using the shorter table:

$P$	$\sim P$	$P \Leftrightarrow (\sim P)$
T	F	F
F	T	F

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(24) Michael says he has made an interesting discovery. Have you?

(25) Mark says that the column for

$$\sim (P \text{ and } Q)$$

is exactly the same as the column for

$$(\sim P) \text{ or } (\sim Q).$$

What do you think?

(26) Eileen listened to what Mark said, and wrote

$$[\sim (P \text{ and } Q)] \Leftrightarrow [(\sim P) \text{ or } (\sim Q)].$$

The class named this "Eileen's formula." Can you make up any other formulas the way Mark and Eileen made up this one?

(24) We are referring, of course, to the realization that some columns in the truth table are identical with others.

(25) Mark is correct.

Incidentally, if your children already know the *distributive law*,

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

(which is sometimes also called "the law for distributing multiplication over addition"), they may notice that

$$\sim (P \text{ and } Q) \Leftrightarrow (\sim P) \text{ or } (\sim Q)$$

is a kind of law for distributing  $\sim$  over *and*. The rule is: prefix a  $\sim$  to each statement, and change *and* to *or*.

(26) Eileen is correct.

As we have seen, there are many more formulas of this type.

## LOGIC (BY THINKING LIKE A MATHEMATICIAN)

In order to understand the point of this chapter, it is worthwhile looking at how *arithmetic* develops as a branch of mathematics. The origins of arithmetic lie, presumably, in general "life experiences" with counting sheep or children or wives or enemies or whatever. After a time, however, *abstractions* are created—such as the number 1, the number 2, the number 3, and so on—which are able to stand by themselves as abstract concepts. We do not need to refer back to the experience from which these concepts were drawn. When we say, abstractly, that

$$2 + 3 = 5,$$

we do not need to ask "five what?" It does not matter whether we are adding avocados or artichokes or kumquats or none of these. In this sense we have eliminated the "meaning" from arithmetic.

When (as in *Discovery*) we develop an axiomatic approach to arithmetic and algebra, we go even further in "eliminating meaning." The phrase "eliminating meaning" is used (unfortunately) quite differently in mathematics and in education. In education, *meaning* is used to refer to *intuition* and *heuristic*. Naturally, we do not wish to eliminate these. In mathematics the word *meaning* is used to refer to something else—namely formal dependence upon the past experiences from which abstract concepts have been drawn. We do *not* wish to be confined to such dependence. Would numbers be equally useful if we had one set of numbers for artichokes, another set of numbers for use in dealing with avocados, and yet a third set of numbers to be used in counting apples, and so on? Quite evidently not; such "meaningful" numbers would have *too much* meaning; they would be too highly *specific*; they would lack generality. The abstract number 2 gains part of its value from its very broad generality; it can refer to two of anything whatsoever, without restriction.

We now wish to take our truth tables, which were born of our study of how our friends used the English language, and get for these ideas an "abstract" existence that will free them from the need to refer to specific English verbal behavior. We can do this by using three ideas: sets, Cartesian products, and mappings.

We shall do this as follows. First, we shall forget about the "meanings" of "true" and "false," and shall consider, instead, a set  $V$  which contains two abstract elements, T and F:

$$V = \{T, F\}.$$

Now, we can construct the Cartesian product (see Chapter 2) of  $V$  with itself:

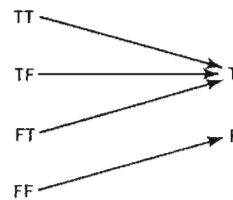
$$V \times V = \{(T, T), (T, F), (F, T), (F, F)\}.$$



Actually, we shall find it more convenient to display the set  $V \times V$  as a vertical column:

$V \times V$   
T T  
T F  
F T  
F F

Finally, we shall consider mappings of  $V \times V$  into  $V$  such as, for example, this one:



Let's now see how this will work out.



CHAPTER 14

**Logic (by thinking like a mathematician)**

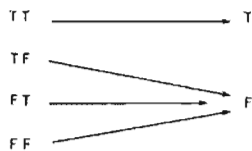
[page 44]

(1) Randy made up a truth table for "and," like this:

		square and triangle
$P$	$Q$	$P$ and $Q$
T	T	T
T	F	F
F	T	F
F	F	F

Did he use "and" the way we have agreed to?

(2) Lex's father says that the relationships in Randy's table can be shown by using what mathematicians call a mapping:



**ANSWERS AND COMMENTS**

(1) **Yes, he did.**

(2) **Yes, Lex's father is correct.**

He says that TT means that  $P$  is true and  $Q$  is true, and that mathematicians call T the image of TT. Do you agree?

(3) Debbie says that, if  $V = \{T, F\}$ , then

$$\{TT, TF, FT, FF\}$$

is just the Cartesian product

$$V \times V.$$

She says Randy has a mapping of  $V \times V$  into  $V$ .

$$V \times V \longrightarrow V$$

What do you think?

(4) How many different ways can you map

$$V \times V \longrightarrow V?$$

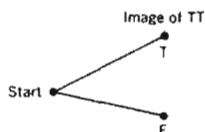
(5) When they want to count something, mathematicians sometimes make a special kind of a drawing which is known as a tree or a tree diagram. Geoffrey tried to count the mappings of  $V \times V$  into  $V$  by drawing a tree diagram.

Geoff says you can map the

TT

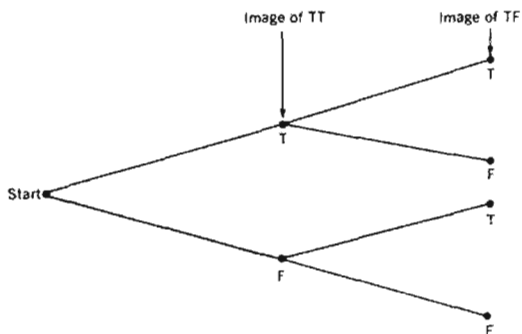
either into T or into F. To show these two choices, he started his tree diagram like this:

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Can you finish Geoff's tree?

(6) After you've mapped TT, Allen says you can map TF either into T or into F:



Do you know what Allen means? Can you finish this tree?

(3) **Debbie is right.**

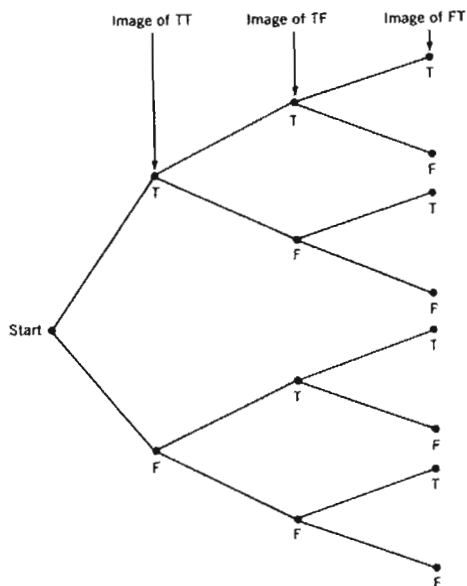
(4) **There are 16 possible mappings.**

The children may not be able to figure this out at first; in questions 5 through 11 we shall develop a systematic method (using trees) to count the possible mappings of  $V \times V$  into  $V$ .

(5) **Geoff has made a good beginning. Your students may be able to finish Geoff's "tree" immediately; if not, we shall work it out, step by step, in questions 6 through 8.**

(6) **Allen is continuing correctly.**

(7) Nancy says that after you've mapped TT and TF, you can map FT either into T or into F:

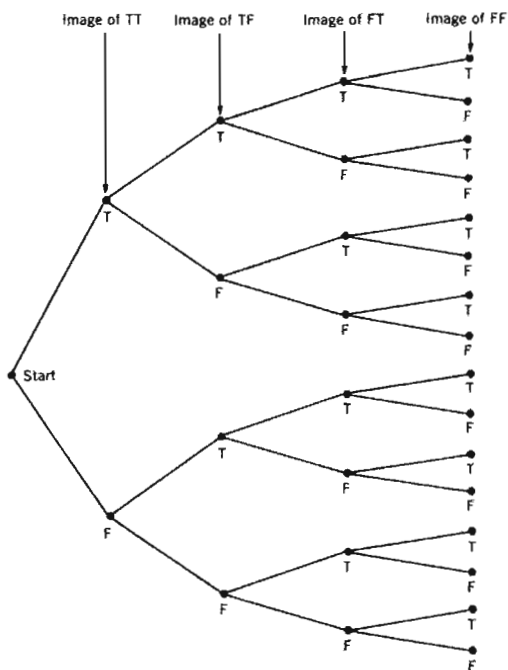


(7) Nancy is continuing the tree correctly.

What do you think?

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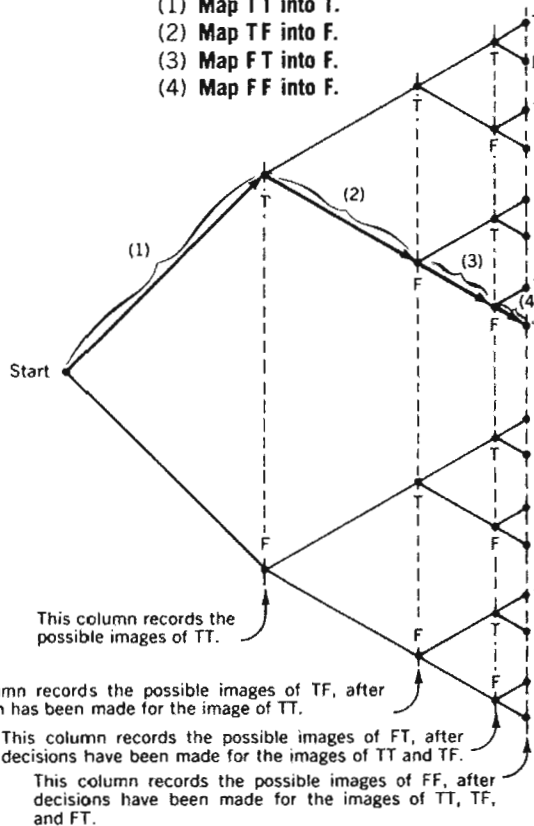
(8) Amy finished the tree like this:



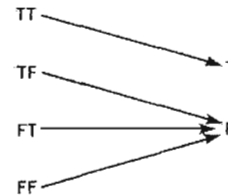
Can you trace a path through Amy's tree that will correspond to the mapping of "and"?

(8) Amy's tree is correct (and the tree is now finished). For the mapping of *and*, we trace the following path.

- (1) Map TT into T.
- (2) Map TF into F.
- (3) Map FT into F.
- (4) Map FF into F.

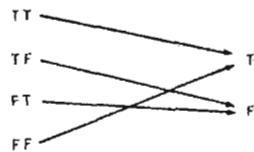


Notice that one can think of this as if we had a rat running through a psychologist's laboratory maze. In the present example, at the "first decision point" the rat chose the left branch of the maze. At the next decision point, the rat chose the right branch; at the third decision point, he again chose the right branch; finally, at the fourth and last decision point, the rat again chose the right branch. This particular route through the maze corresponds to the mapping



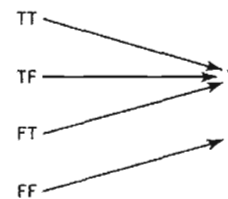
which, in turn, corresponds to the English linguistic behavior that we have represented (using variables  $P$  and  $Q$ ) as " $P$  or  $Q$ ."

(9) Bill represented the mapping of "or" (meaning the "inclusive or") with an arrow diagram:

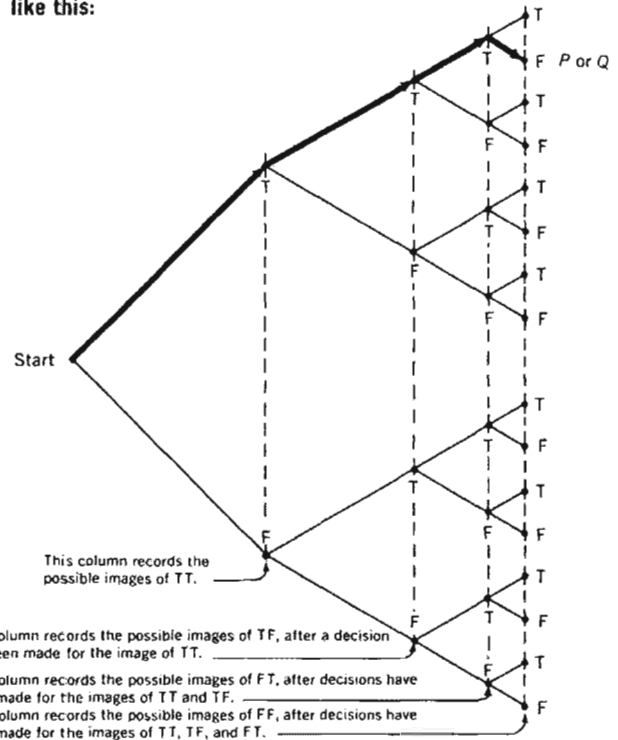


Do you agree?

(9) Bill's diagram is wrong. The correct diagram for " $P$  or  $Q$ " looks like this:



We could also represent " $P$  or  $Q$ " as a "trip through the maze" like this:



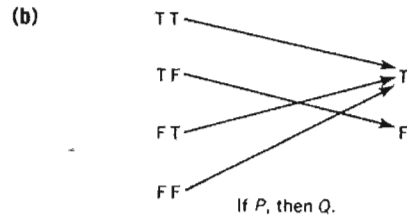
(10) Can you make a diagram of a mapping of

$$V \times V \rightarrow V$$

to correspond to each of the following mappings?

- (a) or
- (b) If  $P$ , then  $Q$ .
- (c)  $P \Leftrightarrow Q$
- (d)  $\sim P$
- (e)  $\sim Q$
- (f)  $(\sim P) \vee (\sim Q)$
- (g)  $(\sim P) \wedge (P)$

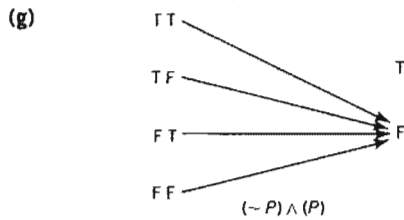
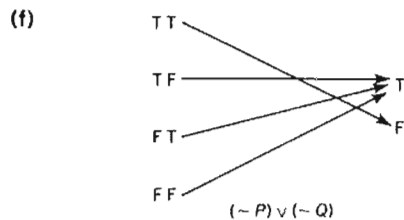
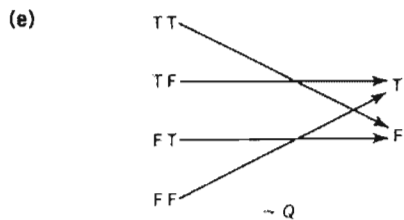
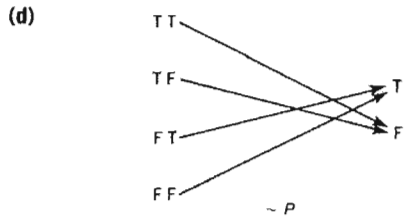
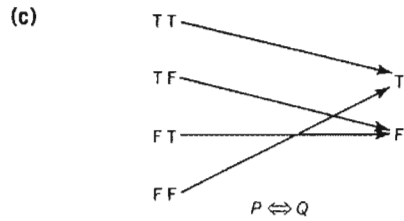
(10) (a) See answer to question 9.



We can introduce some commonly used symbols, as follows:

Symbol	Meaning
$P \vee Q$	$P$ or $Q$ ("inclusive or")
$P \wedge Q$	$P$ and $Q$
$P \Rightarrow Q$	If $P$ , then $Q$ .
$\sim P$	Not $P$
$P \Leftrightarrow Q$	$P$ has the same truth value as $Q$ .

Consequently, the diagram above might be labeled  $P \Rightarrow Q$ .

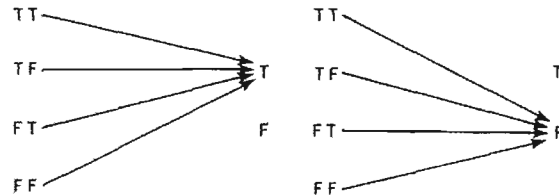


(11) How many possible mappings of  $V \times V \rightarrow V$  are there? Can you show each of them by an arrow diagram?

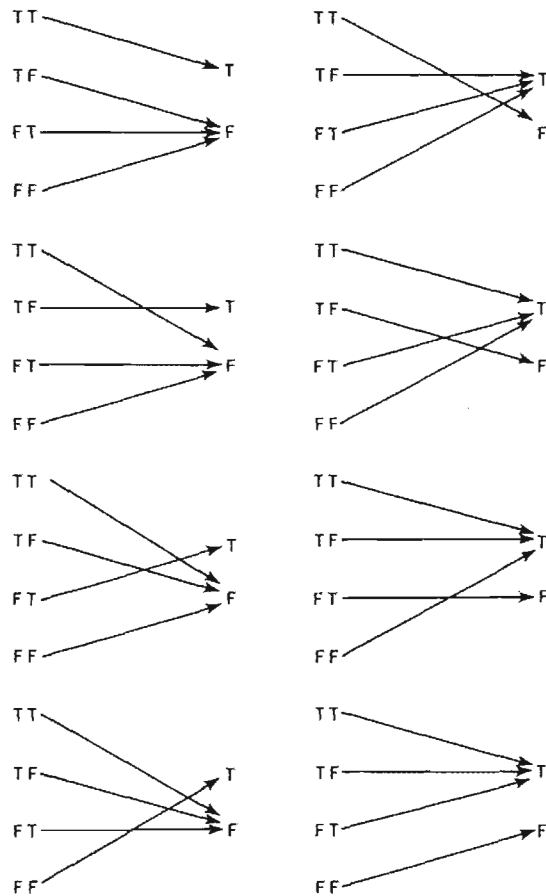
(12) Can you find names for each mapping of  $V \times V \rightarrow V$ ?

(11) and (12) We shall answer these two questions simultaneously, and by two different methods.

First, using diagrams to show mappings of  $V \times V \rightarrow V$ , there are two possible diagrams where all four elements of  $V \times V$  map into the same element of  $V$ :

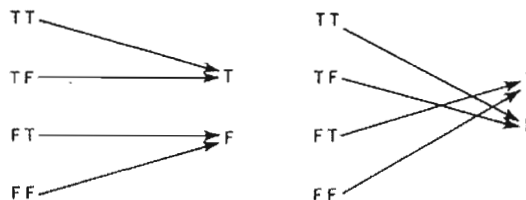


We might call these "4 - 0" diagrams. There are also "3 - 1" diagrams, and "2 - 2" diagrams. How many of each? Evidently, there are eight possible "3 - 1" diagrams; once we choose the "1" mapping, the other three are determined; for each of the four elements of  $V \times V$ , there are two possible images in  $V$ :

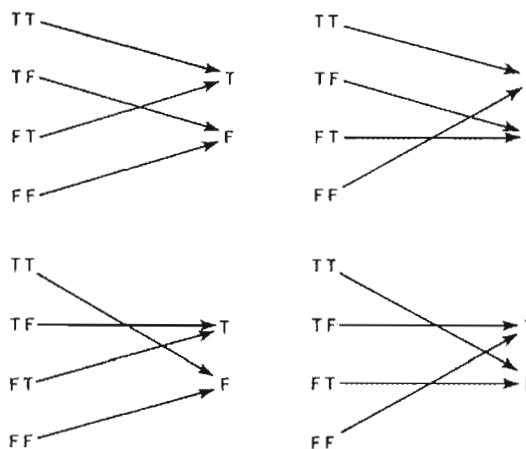


Finally, how many "2 - 2" mappings are there? For TT, we have two choices in selecting an image (i.e., either T or F); for each of these choices, we have two choices for TF. Now, if TT and TF have been mapped into the same image, there are no

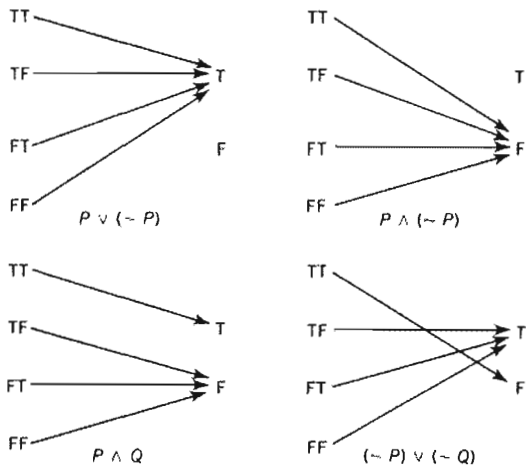
further "free" choices: both FT and FF must map into the other image in order to yield a "2 - 2" mapping. Of this type, we then have two mappings:

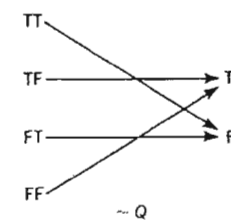
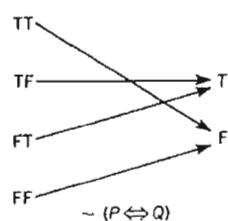
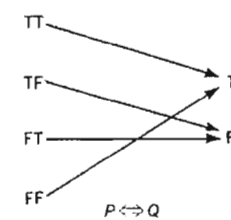
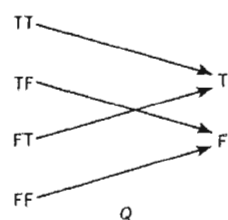
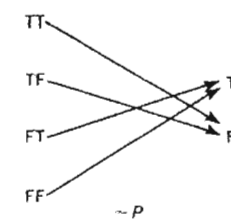
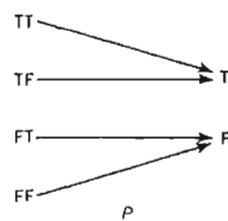
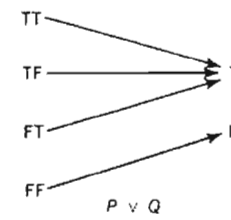
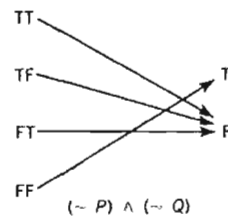
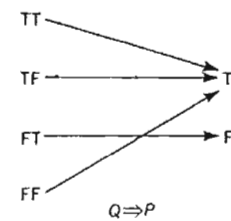
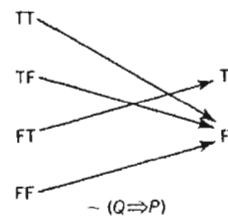
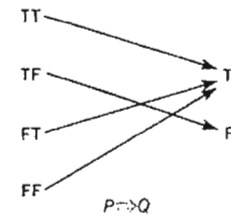
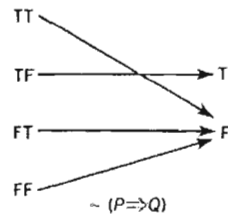


If, however, TT and TF have been mapped into different images, then we have a "free" choice for FT, but FF is thereafter determined:



Now, can we find "names" for each of these mappings, using a vocabulary of the following symbols:  $P, Q, \sim, \vee, \wedge, \Rightarrow, \Leftrightarrow$ , and parentheses? The answer is that we can. We present below the 16 possible pictures, each with an appropriate name. *Many other names are possible for each diagram.*

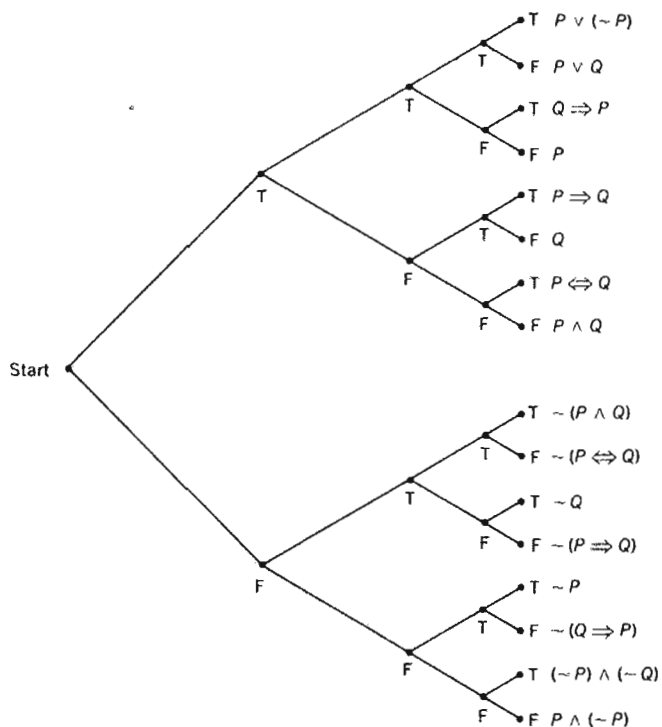




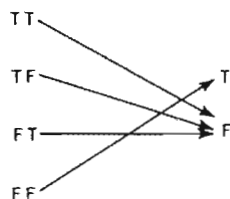
There is a second abstract approach; instead of using diagrams to show mappings of  $V \times V$  into  $V$ , we can use a "tree"



picture, to show in one diagram all 16 possible mappings. Here is such a diagram, labeled with appropriate names:



There is much more that can be done with this kind of question and with this kind of concept. For example, we can seek other names for each of these mappings. The mapping



can be named  $(\sim P) \wedge (\sim Q)$ ; however, this same mapping can evidently be named  $\sim (P \vee Q)$ .

Our students usually discover this for themselves; this discovery leads us to write

$$[(\sim P) \wedge (\sim Q)] \Leftrightarrow [\sim (P \vee Q)].$$

Some of our students, familiar with the *distributive law* of arithmetic (see *Discovery*, Chapters 29 and 32),

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla),$$

have called the statement

$$\sim (P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q)$$

“a kind of distributive law,” or a “law for the distribution of  $\sim$  over  $\vee$ .” This is a rather insightful description.

There is another aspect which our students often enjoy. We have succeeded in naming every one of the 16 mappings of  $V \times V \rightarrow V$ . To do this, we have used a "vocabulary" consisting of  $P, Q, \neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$ , together with parentheses. The resulting names were not unique. This raises several further questions immediately: Could we use a smaller vocabulary, and still be able to name all 16 mappings? What is the minimal vocabulary that will suffice? How large is this minimal vocabulary? If we use the minimal vocabulary, will the names then be unique?

There is still another direction for further exploration. We have used a two-valued logic—that is to say, our "truth-value space"  $V$  contains two elements. How would all of this work if  $V$  contained three or more elements?

As observed on the pages of a book, this kind of mathematics can look remote and formidable. Within our own experience, a teacher who will explore these ideas with his students can find much that is intriguing, fun, and actually quite accessible. Good luck in your exploring!



## INFERENCE SCHEMES

Inference schemes play an important role in the application of *mathematical logic* to high school mathematics—for example, in the use of logical language in discussing algebra and geometry.



### CHAPTER 15

## Inference Schemes

[page 47]

In several preceding chapters, we have looked at logic from the point of view of the logical connectives that commonly occur *within* sentences: connectives such as “and,” “or,” “not,” “if . . . , then . . . ,” and so on.

We now wish to look at the logical relations that often exist *between* sentences. Here are some examples:

(1) If Mr. Wilson is the guilty person, then he certainly had to be in New York City on July 10, 1967. However, Mr. Wilson was *not* in New York City on July 10, 1967. Therefore, Mr. Wilson cannot be the guilty person.

Jerry has tried to take these statements about Mr. Wilson, and represent them as an inference scheme:

Jerry lets  $P$  stand for “Mr. Wilson is the guilty person.”

He lets  $Q$  stand for “Mr. Wilson was in New York City on July 10, 1967.”

Can you now represent the statements about Mr. Wilson, using Jerry’s  $P$  and  $Q$ ?

## ANSWERS AND COMMENTS

(1) The statements could be represented like this:

Mr. Wilson is the guilty person.	$P$
He was in New York City on July 10, 1967.	$Q$
If Mr. Wilson is the guilty person, then he was in New York City on July 10, 1967.	$P \Rightarrow Q$
He was not in New York City on July 10, 1967.	$\sim Q$
Therefore, he cannot be the guilty person	$\sim P$

Putting all of this together, as in the argument in question 1, we get:

$$\begin{array}{r} P \Rightarrow Q \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

A pattern such as this is called an inference scheme. In *using* these patterns, we have two tasks:

- (i) to translate from words to letters, as we have just done;
- (ii) to determine whether the inference scheme is *valid*, or not.

*Testing for Validity.* There are various ways for testing whether an inference scheme is *valid* or not. We shall now test the scheme

$$\begin{array}{r} P \Rightarrow Q \\ \sim Q \\ \hline \therefore \sim P \end{array}$$

by the method of *using a truth table*:

$P$	$Q$	$P \Rightarrow Q$	$\sim Q$	$\sim P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Since we are told that the statement  $P \Rightarrow Q$  is *true*, we know that, whatever truth values  $P$  and  $Q$  may have, we *cannot* be in the second row of the truth table:

$$\begin{array}{c} \rightarrow P \Rightarrow Q \leftarrow \\ \hline \sim Q \\ \hline \therefore \sim P \end{array}$$

$P$	$Q$	$P \Rightarrow Q$	$\sim Q$	$\sim P$
T	T	T	F	F
<del>T</del>	<del>F</del>	<del>F</del>	<del>T</del>	<del>F</del>
F	T	T	F	T
F	F	T	T	T

Since we are told that the statement  $\sim Q$  is *true*, we know that we cannot be in the first or third rows of the truth table:

$$\begin{array}{c} P \Rightarrow Q \\ \rightarrow \sim Q \leftarrow \\ \hline \therefore \sim P \end{array}$$

$P$	$Q$	$P \Rightarrow Q$	$\sim Q$	$\sim P$
<del>T</del>	<del>T</del>	<del>T</del>	<del>F</del>	<del>F</del>
<del>T</del>	<del>F</del>	<del>F</del>	<del>T</del>	<del>F</del>
F	T	T	F	T
F	F	T	T	T

We are now ready to see if  $\sim P$  is a "legitimate" and *necessary* conclusion. We do this by scanning down the column headed  $\sim P$ , to see if there are any F's remaining:

$$\begin{array}{c} P \Rightarrow Q \\ \sim Q \\ \hline \therefore \sim P \leftarrow \end{array}$$

$P$	$Q$	$P \Rightarrow Q$	$\sim Q$	$\sim P$
<del>T</del>	<del>T</del>	<del>T</del>	<del>F</del>	<del>F</del>
<del>T</del>	<del>F</del>	<del>F</del>	<del>T</del>	<del>F</del>
<del>F</del>	<del>T</del>	<del>T</del>	<del>F</del>	<del>T</del>
F	F	T	T	T

Since only T's remain in this column, *the inference is valid.*

It is worth pausing a moment to see what, if anything, we have accomplished. Our work here has been *purely formal*—that is to say, it has depended *only upon the form of the argument*. Obviously, by purely formal means we cannot arrive at the "truth"—that is, in the present example, we cannot establish whether or not Mr. Wilson really was in New York City on July 10, 1967. This is not a *formal* matter, it is a statement about *reality*. It cannot be settled by truth tables or other formal means, but must be established by the testimony of witnesses. The truthfulness of witnesses is not a matter of logic.

What we *have* done is to show that the *form* of the argument in this example is legitimate. Consequently, we can conclude that *if the witnesses are telling the truth* in their various separate statements, then it is legitimate to conclude that the combined impact of their separate statements establishes Mr. Wilson's innocence.

There are many aspects to "reasoning" and "judging," and formal logical inference is only a small part of what is involved—but

even a small part can be important. We should not over-value formal logic, nor under-value the other parts of the process of "judging" or "reasoning" – other parts, that is, such as intuitive assessment of credibility, plausible inference, probabilities, and so on.

I suspect that many youngsters reject mathematics because their elders made exaggerated claims for it, which inevitably resulted in disillusionment. We surely want to avoid making exaggerated claims for formal logic. It is a small piece of the machinery for seeking "truth," but it is an important one.

(2) Marie says the statement, "If Mr. Wilson is the guilty person, then he was in New York City on July 10, 1967," can be represented as

$$P \Rightarrow Q.$$

What do you think?

(3) Nancy says the statement, "Mr. Wilson was not in New York City on July 10, 1967," can be represented as

$$\sim Q.$$

What do you think?

(4) Al says the whole discussion about Mr. Wilson can be represented this way:

$$\frac{P \Rightarrow Q, \sim Q}{\sim P}$$

Do you see how Al's notation works?

(5) Consider these statements:

If Jerry believes that smoking causes cancer, then he would be foolish to smoke. Jerry does believe that smoking causes cancer. Therefore, Jerry would be foolish to smoke.

Can you write out the inference scheme that seems to be used here?

(2) Marie is correct.

(3) Nancy is correct.

(4) Either of these notations is in common use:

$$\frac{P \Rightarrow Q, \sim Q}{\sim P} \quad \text{or} \quad \frac{P \Rightarrow Q}{\sim Q} \therefore \sim P$$

(5) We might do it like this:

Jerry believes that smoking causes cancer.  $P$   
 Jerry would be foolish to smoke.  $Q$   
 If Jerry believes that smoking causes cancer, then Jerry would be foolish to smoke.  $P \Rightarrow Q$

$$\frac{P \Rightarrow Q}{P} \therefore Q$$

This is one of the most important of all inference schemes. It was known by the ancient Greeks; today it is one of the foundations for modern logic. It goes, incidentally, under the name of *modus ponens*. We can check the validity of *modus ponens* by using a truth table, as follows:

$P$	$Q$	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Since we are told that the statement  $P \Rightarrow Q$  is true, we know we *cannot* be in the second row of the truth table:

$$\begin{array}{c} \rightarrow P \Rightarrow Q \leftarrow \\ \hline P \\ \hline \therefore Q \end{array}$$

$P$	$Q$	$P \Rightarrow Q$
T	T	T
<del>T</del>	<del>F</del>	<del>F</del>
F	T	T
F	F	T

Since we are told that  $P$  is true, we know we *cannot* be in the third or fourth rows of the truth table:

$$\begin{array}{c} \rightarrow P \Rightarrow Q \leftarrow \\ \hline P \\ \hline \therefore Q \end{array}$$

$P$	$Q$	$P \Rightarrow Q$
T	T	T
<del>T</del>	<del>F</del>	<del>F</del>
<del>F</del>	T	T
<del>F</del>	F	T

Now, we check whether  $Q$  is a valid conclusion by scanning down the column headed  $Q$  to see if any F's remain.

$$\begin{array}{c} \rightarrow P \Rightarrow Q \leftarrow \\ \hline P \\ \hline \therefore Q \end{array}$$

$P$	$Q$	$P \Rightarrow Q$
T	T	T
<del>T</del>	<del>F</del>	<del>F</del>
<del>F</del>	T	T
<del>F</del>	F	T

Since only T's remain in this column, the *conclusion is valid*. (That is, the statement  $Q$  must be true, if the other statements really are true.)

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(6) Consider these statements:

If Mr. Harper was in San Diego at 10 A.M., Tuesday, then he must be innocent. If he was driving from Los Angeles to San Diego at 10 A.M., Tuesday, then he must be innocent. Now, we know definitely that he was either in San Diego at 10 A.M., Tuesday, or else he was driving from Los Angeles to San Diego at that time. Therefore, Mr. Harper must be innocent.

Can you write out the inference scheme that seems to be used here?

(6) We can translate into letters as follows:

- Mr. Harper was in San Diego at 10 A.M. Tuesday.  $P$
- Mr. Harper is innocent.  $Q$
- If Mr. Harper was in San Diego at 10 A.M., Tuesday, then he must be innocent.  $P \Rightarrow Q$
- Mr. Harper was driving from Los Angeles to San Diego at 10 A.M., Tuesday.  $R$
- If Mr. Harper was driving from Los Angeles to San Diego at 10 A.M., Tuesday, then he is innocent.  $R \Rightarrow Q$
- Either Mr. Harper was in San Diego at 10 A.M., Tuesday, or else he was driving from Los Angeles to San Diego at 10 A.M., Tuesday.  $P \vee R$

We can now represent the argument by the following inference scheme:

$$\begin{array}{l} P \Rightarrow Q \\ R \Rightarrow Q \\ \hline P \vee R \\ \hline \therefore Q \end{array}$$

Now, we can check the *validity* of this inference scheme by using truth tables as follows:

$P$	$Q$	$R$	$P \Rightarrow Q$	$R \Rightarrow Q$	$P \vee R$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	T	T	F	T
T	T	F	T	T	T
T	F	F	F	T	T
F	T	F	T	T	F
F	F	F	T	T	F

Since we are told that the statement  $P \Rightarrow Q$  must be *true*, we know that we *cannot* be in rows 2 or 6 of the truth table:

$$\begin{array}{l} \rightarrow P \Rightarrow Q \leftarrow \\ R \Rightarrow Q \\ \hline P \vee R \\ \hline \therefore Q \end{array}$$

$P$	$Q$	$R$	$P \Rightarrow Q$	$R \Rightarrow Q$	$P \vee R$
T	T	T	T	T	T
<del>T</del>	<del>F</del>	<del>T</del>	<del>F</del>	<del>F</del>	<del>T</del>
F	T	T	T	T	T
F	F	T	T	F	T
<del>T</del>	<del>T</del>	<del>F</del>	<del>T</del>	<del>T</del>	<del>T</del>
T	F	F	F	T	T
F	T	F	T	T	F
F	F	F	T	T	F

Since we are told that the statement  $R \Rightarrow Q$  must be *true*, we know we *cannot* be in rows 2 or 4 of the truth table:

$$\begin{array}{l} P \Rightarrow Q \\ \rightarrow R \Rightarrow Q \leftarrow \\ \hline P \vee R \\ \hline \therefore Q \end{array}$$

$P$	$Q$	$R$	$P \Rightarrow Q$	$R \Rightarrow Q$	$P \vee R$
T	T	T	T	T	T
<del>T</del>	<del>F</del>	<del>T</del>	<del>F</del>	<del>F</del>	<del>T</del>
F	T	T	T	T	T
<del>F</del>	<del>F</del>	<del>T</del>	<del>T</del>	<del>F</del>	<del>T</del>
T	T	F	T	T	T
<del>T</del>	<del>F</del>	<del>F</del>	<del>F</del>	<del>T</del>	<del>T</del>
F	T	F	T	T	F
F	F	F	T	T	F

Since we are told that the statement  $P \vee R$  must be true, we know we cannot be in rows 7 or 8 of the truth table:

$$\begin{array}{l} P \Rightarrow Q \\ R \Rightarrow Q \\ \rightarrow \underline{P \vee R} \leftarrow \\ \therefore Q \end{array}$$

P	Q	R	$P \Rightarrow Q$	$R \Rightarrow Q$	$P \vee R$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	T	T	F	T
T	T	F	T	T	T
T	F	F	F	T	T
F	T	F	T	T	F
F	F	F	T	T	F

Finally, to test the legitimacy of drawing the conclusion  $Q$  from this data, we have merely to scan down the column headed  $Q$  and see if any F's remain:

$$\begin{array}{l} P \Rightarrow Q \\ R \Rightarrow Q \\ \underline{P \vee R} \\ \rightarrow \therefore Q \leftarrow \end{array}$$

P	Q	R	$P \Rightarrow Q$	$R \Rightarrow Q$	$P \vee R$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	T	T	F	T
T	T	F	T	T	T
T	F	F	F	T	T
F	T	F	T	T	F
F	F	F	T	T	F

Since only T's remain, the inference is valid.

(7) Can you make up any inference schemes of your own that seem to be valid?

(7) Here are some inference schemes commonly suggested, some of which are valid and some of which are not:

$$\begin{array}{l} P \rightarrow Q \\ \underline{Q} \\ \therefore P \quad (\text{not valid}) \end{array} \quad \begin{array}{l} P \rightarrow (\sim Q) \\ \underline{Q} \\ \therefore \sim P \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \vee Q \\ \underline{\sim Q} \\ \therefore P \quad (\text{valid}) \end{array}$$

$$\begin{array}{l} P \vee Q \\ \underline{\sim P} \\ \therefore Q \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \wedge Q \\ \underline{P} \\ \therefore P \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \Rightarrow Q \\ Q \Rightarrow R \\ \underline{\sim R} \\ \therefore \sim P \quad (\text{valid}) \end{array}$$

$$\begin{array}{l} P \Rightarrow Q \\ Q \Rightarrow R \\ \underline{P} \\ \therefore R \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \vee Q \vee R \\ \underline{\sim P} \\ \underline{\sim Q} \\ \therefore R \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \vee (Q \wedge R) \\ \underline{\sim Q} \\ \therefore P \quad (\text{valid}) \end{array}$$

$$\begin{array}{l} P \vee (Q \wedge R) \\ \underline{\sim P} \\ \therefore Q \quad (\text{valid}) \end{array} \quad \begin{array}{l} P \Rightarrow Q \\ Q \Rightarrow R \\ \underline{R \Rightarrow S} \\ \therefore P \Rightarrow S \quad (\text{valid}) \end{array}$$



- (8) Toby made up this inference scheme:

$$\frac{P \Rightarrow Q, Q \Rightarrow R}{P \Rightarrow R}$$

Do you think it is valid? Can you give some examples, using words?

- (9) Can you find a way to test the truth of inference schemes by using truth tables?

What good would such a method be?

- (10) Sarah made up this inference scheme:

$$\frac{P \Rightarrow Q, \sim P}{\sim Q}$$

Do you think it is valid? Can you give some examples, using words? Can you test it by using a truth table?

- (11) How many valid inference schemes can you list?

(12) Who might be interested in studying inference schemes? Do you think mathematicians would? Do you think logicians would? Do you think lawyers would? Who else might be? What good would it do to study inference schemes?

- (8) Yes, it is valid. Here is one example, with words:

If I go to Boston, I'll buy a recorder.  
If I buy a recorder, I'll learn to play it.  
Therefore, if I go to Boston, I'll learn to play the recorder.

Note that, in our "vertical" notation, Toby's inference scheme would be written:

$$\frac{P \Rightarrow Q}{Q \Rightarrow R} \therefore P \Rightarrow R$$

- (9) We have discussed this extensively in the preceding questions.

One reason why our test of validity (by using truth tables) is valued by some people is that it is a step—a rather small step—in the direction of "objectivity," which has been a common goal in academic life in the twentieth century. By "objectivity" people seem to mean that anyone is compelled to agree, or that any well-educated person would necessarily arrive at the same conclusion. This is at best an elusive ideal, and at worst a dubious one—but truth tables do provide a tool that "we can all use the same way."

- (10) Sarah's inference scheme is not valid. You can prove this by using a truth table.

- (11) Obviously, there are a great many.

- (12) Lawyers, mathematicians, and logicians probably all have something to gain from the study of inference schemes. So do psychologists, teachers, linguistics experts, and anyone else interested in understanding how humans reason.

One might say that inference schemes play a role (although a small one) in the task of building an *explicit* description of how people reason. However, the goal of being explicit is not without its own hazards.

You are encouraged to read the beautiful essay by Aldous Huxley entitled *The Education of an Amphibian*, and also his essay *Education on the Nonverbal Level* [see Huxley (43), (44)].

## THE GAME OF CLUES

The Game of Clues is actually a modified version of the game Hidden Numbers which was introduced by Professor David Page of the University of Illinois.\* This game is always fun for the students, but makes some demands upon the teacher.

The version of the game described in the *Student Discussion Guide* is the final, sophisticated version. Before the students read this chapter, you may want to prepare them by playing one or more simpler versions. Matters will be made simpler by having the teacher take the role of the TWS team and by having the entire class take the role of the DISC team. We shall assume that you do this. The simplest version is to make all clues true, and omit rules 6 through 14.

A slightly more complicated version is to use all rules, except 8, 9, and 11. That is to say, do not require DISC team to make a careful, explicit list of which clues they are using when they claim a contradiction. After using this version, you can introduce the requirement that the DISC team make careful, explicit lists of precisely which clues are involved in a contradiction; you are then playing the first, sophisticated version as it is described in the *Student Discussion Guide*.

Why do we play this game? There are many reasons, but perhaps the most important are these: We want to give the children experience with such mathematical ideas as *implication*, *contradiction*, and *uniqueness*. This game gives us a lesson format within which we can do a very broad range of mathematics – including “review” of arithmetic – which is new, exciting, and fun.



### CHAPTER 16

## The Game of Clues

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The rules for the game of clues are as follows:

One team (or one person) has a secret. Let's call this team TWS, for “team with secret.” The other team seeks to discover this secret. Let's call this team DISC, for “discovery.”

1. TWS writes some numbers on a piece of paper which then is sealed in an envelope, or otherwise put where it cannot be read. (For example, someone can fold the paper and sit on it.)

2. DISC seeks to force TWS to disclose the “secret” numbers, and to let everyone read the paper.

3. Only positive integers are allowed. Repetitions are allowed; for example, the secret numbers might be:

1, 3, 5, 7, 7, 7, 7.

## ANSWERS AND COMMENTS

\*At the time of this writing, Professor Page is working at the important institution known as Educational Services, Incorporated, or “ESI” for short. You may want to read the Quarterly Reports published by ESI, which can be obtained by writing to: Educational Services Incorporated, 55 Chapel Street, Newton, Massachusetts 02158.

4. In guessing the secret numbers, DISC does not have to guess the order in which they are written; for example,

7, 3, 5, 7, 1, 7, 7

would count as the same list as the one given in the rule preceding.

5. TWS writes clues on the board, labeling the clues *a, b, c, ...*, and so on (it is desirable to omit "F" and "T" as labels, since we have a different use for them).

6. The clues may be true or they may be false.

7. Anytime that DISC believes there is a contradiction in a certain set of clues, DISC lists the clues in question and tries to show that there is a contradiction in these clues.

8. DISC is right about the contradiction if the clues they list do contain a contradiction, and if no proper subset of the clues on the list contains a contradiction.

9. DISC is wrong about the contradiction if the clues they list do not contain a contradiction or if a proper subset of the clues does contain a contradiction.

10. At the start of the game, DISC has 5 points.

11. Anytime DISC is wrong about a contradiction, it loses one point.

12. Anytime DISC is right about a contradiction, TWS must mark T (for true) or F (for false) beside each clue that is involved in the contradiction. TWS must be correct in marking T's and F's (even though TWS is allowed to make some of the clues themselves false).

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13. The game ends in one of two ways: If DISC loses all 5 points, then TWS tears up the secret paper and never allows it to be read (DISC has "lost"). If, on the other hand, DISC is able to force disclosure of the paper, then everyone on the DISC team is allowed to read it, and DISC has "won."

14. The procedure by which DISC may be able to force disclosure of the secret is this: whenever it believes it is in a position to do so, DISC can list the numbers that it believes must be written on the paper, and can bet TWS that no other collection of numbers would satisfy all the known truth values of the clues. (That is, no other collection of numbers would make true statements of all the clues labeled T and false statements of all the statements labeled F.) If TWS can find any other collection of numbers that will be consistent with the T's and F's, then DISC loses the bet, and DISC's points are reduced to zero. (Which, of course, means the secret paper is torn up and the numbers never disclosed.)

If TWS cannot find any other collection of numbers that will be consistent with the indicated T's and F's, then DISC wins the bet, and TWS is forced to disclose the secret.

In order to make the game interesting, TWS must provide a growing collection of interesting clues.

Here is a sample game:

DISC begins, of course, with 5 points.

TWS begins by listing these clues:

- a. 5 numbers on paper.
- b. All odd numbers.
- c. Their sum is 26.
- d. The largest number is 7.
- e. The smallest number is 8.

DISC says there is a contradiction in clues  $a$ ,  $b$ , and  $c$ , because an odd number of odd numbers cannot add up to an even total.

Since DISC is right about

$$\{a, b, c\},$$

it is necessary for TWS to label  $a$ ,  $b$ , and  $c$  as either T or F; TWS does this as follows:

- F a. 5 numbers on paper.
- T b. All odd numbers.
- F c. Their sum is 26.
- d. The largest number is 7.
- e. The smallest number is 8.

TWS changes the clues to look like this:

- a. 7 numbers on paper.
- T b. All odd numbers.
- c. Their sum is 12. [page 51]
- d. The largest number is 7.
- e. The smallest number is 8.

DISC says that

$$\{a, b, c\}$$

still contains a contradiction: an odd number of odd numbers cannot add up to an even sum.

Since DISC is right about this contradiction, TWS must label  $a$ ,  $b$ , and  $c$  as T or F. They do this as follows:

- T a. 7 numbers on paper.
- T b. All odd numbers.
- F c. Their sum is 12.
- d. The largest number is 7.
- e. The smallest number is 8.

DISC says that

$\{d, e\}$

contains a contradiction, because the largest number cannot be smaller than the smallest number.

Since DISC is right about this, TWS must mark T's and F's on

$\{d, e\}$ .

They do this as follows:

- T a. 7 numbers on paper.
- T b. All odd numbers.
- F c. Their sum is 12.
- T d. The largest number is 7.
- F e. The smallest number is 8.

TWS changes the clues to read like this:

- T a. 7 numbers on paper.
- T b. All odd numbers.
- c. Their sum is 13.
- T d. The largest number is 7.
- F e. The smallest number is 8.

Although they are not yet forced to do so, TWS labels clue c as T, in order to make the game move along faster. The clues now look like this:

- T a. 7 numbers on paper.
- T b. All odd numbers.
- T c. Their sum is 13.
- T d. The largest number is 7.
- F e. The smallest number is 8.

(1) Can you finish this game?

(1) At this point DISC has 5 points and these clues are on the board:

- T a. 7 numbers on paper.
- T b. All odd numbers.
- T c. Their sum is 13.
- T d. The largest number is 7.
- F e. The smallest number is 8.

What do we know from this? Since we have only *odd* numbers, the largest of which is 7, we know that there *is* at least one 7 on the paper and that the other numerals, if any, are 1, 3, and 5.

Now, since there are 7 numerals on the paper, they cannot be too large, or the sum will exceed 13. Let's see, is 7, 1, 1, 1, 1, 1, 1 possible? Yes, since  $7 + 1 + 1 + 1 + 1 + 1 + 1 = 13$ .

But ... if we increase *any* number on the list *the sum will be too large!* Hence, 7, 1, 1, 1, 1, 1, 1 is the *only* possible answer. That is to say, the answer is now *uniquely* determined, as mathematicians would describe it. (Remember, *order* does not count!)

We consider that 1, 1, 1, 7, 1, 1, 1, is "not really different" from 7, 1, 1, 1, 1, 1, 1, and so on.)

The DISC team now bets that this *must* be what is written on the paper. Since DISC is correct, TWS must disclose the hidden paper for all to see.

(2) Why don't you write your own secret numbers, and make up your own clues?

(2) You and your students will want to make up your own clues. Here, however, are a few types of clues that we have found useful:

The sum is 25.

The product is 13.

The numbers are all odd.

The numbers are all even.

The numbers are not all odd.

The numbers are all different.

The smallest number is 3.

The largest number is 21.

The numbers are all prime.

The numbers are all multiples of 7.

The numbers are of the form  $\alpha, \alpha, \alpha, \beta, \beta, \gamma$ , where  
 $\alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma$ .

Two of the numbers, added together, make 12—provided you pick the right two numbers on the paper.

... but it is better to invent your own kinds of clues.

## Part Three ■ Measurement Uncertainties

chapter 17 / Pages 52-55 of Student Discussion Guide

### MEASUREMENT UNCERTAINTIES

Traditionally, it has been all too easy for elementary school children to get the idea that every question has exactly one right answer. Further, they often believe that this right answer is "perfect" and "exact" and so forth. This belief was probably encouraged, at least in part, by the kinds of questions the children encountered.\*

Now, in normal adult life, in matters of business, in matters of art or history, and quite equally in matters of science and mathematics, things are not this simple. Many questions have no answers at all, some have many "right" answers, and some have many answers which are "almost right" — where we can never find an answer that is *exactly* right.

How far is it from the earth to the sun? Obviously, we don't really know, and every attempt to measure this distance will probably produce an "answer" different from all other attempts.

Again, what decimal name — by which we mean, of course, a "terminating" decimal name that you can actually write — is a name for the square root of 2? It can easily be shown that there is none. However, there *are* some decimal names which are reasonably good approximations. If we square 2, we get 4, so 2 is too large. If we square 1, we get 1, so 1 is too small. If we square 1.5, we get 2.25, so 1.5 is too large. If we square 1.4, we get 1.96, so 1.4 is too small. As we continue in this fashion, we find that 1.42 is too large ( $1.42^2 = 2.0164$ ), that 1.41 is too small ( $1.41^2 = 1.9881$ ), and so on . . . However, we shall never find a terminating decimal whose square is *exactly* 2 — that is, we can never find a terminating decimal name for the square root of 2. We can, however, find some *very* good approximations. What is the "right" name? Among terminating decimals, there is none. That's just the way it is.

We need not go to such sophisticated questions as the distance from the earth to the sun. If we try to measure the length of the classroom, we shall find that small errors in measurement are *inevitably* part of our answers. We *cannot find exactly how long the classroom is*. No one can. (In fact, it can even be argued that, when one considers distances as small as millionths of an inch, or smaller, the length of the classroom keeps changing, due to temperature changes, settling of the building, abrasive action on the walls, plaster and paint flaking, and so on.)

Even if we agree that we shall measure from *exactly* this spot here on the front wall to this spot here on the rear wall, we can-

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\*Some new mathematics material, being developed in Great Britain by the Nuffield Foundation Project, under the direction of Geoffrey H. Matthews at St. Dunstan's College, London, makes excellent use of questions which have no answers, questions which have exactly one answer, and so on.

The work cited in Appendix A, Schwab (1), is the finest discussion on this that I have ever seen — and one which you should not miss reading.

not expect that we can repeat the measurement many times and get *exactly* the same answer. In general we cannot do so. (Philosophically it is sometimes important to notice that this question *can* be formulated differently. To every measuring instrument we can assign a "discrimination threshold"—that is, we can try to *guess* to the nearest yard, or to measure to the nearest foot, or (using a finer measuring tool) we can try to measure to the nearest sixteenth of an inch, and so on. If this "discrimination threshold" is large enough, then we *can* get *absolute agreement*—but, of course, this is *not* agreement as to the actual length of the room.)

In the present chapter, we have 10 students, working independently and in secret, *guess* the length of the room. We take the 10 numbers obtained in this way, compute an average, and also perform a computation to see how closely the 10 students agree. Obviously, we expect considerable variation in the 10 answers. We next have 10 students, working independently and secretly, *measure* the length of the room, using 6-inch rulers. These are, of course, awkwardly small rulers for so large a distance; we must move them end-to-end many times before we are done, and each move entails the possibility of some error. Moreover, most 6-inch rulers are far from precision instruments, and may well involve errors in themselves. With the 10 numbers we get this way, we again compute the average, and try to calculate the amount of agreement or disagreement. (Obviously, we *expect* to find more agreement—or less disagreement—than when 10 students guessed; however, we still expect considerable disagreement.)

We next repeat this procedure, using good-quality yardsticks or meter sticks. We expect, in this case, to find somewhat greater agreement. Finally, we go through the same procedure using good-quality tape measures. This time we expect even less disagreement—but we would still expect some disagreement.

By this time, we *hope* the children are coming to realize that every measured "answer" is "wrong"—indeed, *there is no way to find the "right" answer*—but that in some cases the error is probably much smaller than in other cases.

A number of possible refinements can be made. For one thing, you may get better results using a larger distance than the length of the classroom—for example, use instead the length of the school corridor. This, in effect, gives the children greater refinement in their measuring instruments, since there is a tendency for them to use  $\frac{1}{16}$  inch, or  $\frac{1}{32}$  inch, as the smallest distance they will bother to report. If we cannot get into finer discrimination in visually reading meter sticks and rulers, then we can in effect make  $\frac{1}{32}$  inch an "effectively smaller" amount by measuring longer over-all distances.

As another refinement in attempting to determine the amount of disagreement, you may want to use either the *variance* or the *standard deviation*. They are computed as follows:

Suppose the 10 numbers we got from the 10 children were

$$a, b, c, d, e, f, g, h, k, m.$$

Then we find the *average*  $A$  as

$$A = \frac{a + b + c + d + e + f + g + h + k + m}{10}.$$



For each of the original 10 numbers, we find its *deviation* from the average:

Original number	Deviation from average
$a$	$a - A$
$b$	$b - A$
$c$	$c - A$
$d$	$d - A$
$e$	$e - A$
$f$	$f - A$
$g$	$g - A$
$h$	$h - A$
$k$	$k - A$
$m$	$m - A$

Note that these deviations in some cases will be *positive* and in other cases will be *negative*. (In fact, if you add up the "deviations" column, the "positives" and "negatives" should just cancel out, and the total for the column should be zero.)

We now take the 10 "deviations" and square each one:

$$\begin{aligned} &(a - A)^2 \\ &(b - A)^2 \\ &(c - A)^2 \\ &(d - A)^2 \\ &(e - A)^2 \\ &(f - A)^2 \\ &(g - A)^2 \\ &(h - A)^2 \\ &(k - A)^2 \\ &(m - A)^2 \end{aligned}$$

We now add the column of squares, calling the total  $T$ . We now divide this number  $T$  by 10 (i.e., by the number of measurements we started with). The result,  $\frac{1}{10} \times T$ , is called the *variance*. The positive square root of the variance,

$$\sqrt{\frac{1}{10} \times T},$$

is known as the *standard deviation*.

Before teaching this lesson, you may want to view the Madison Project film entitled "Average and Variance." If you wish to study further the ideas of variance and standard deviation, consult Mosteller (83).



## CHAPTER 17

## Measurement Uncertainties

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(1) Can you measure how long your classroom is?

## ANSWERS AND COMMENTS

(1) This question is intended mainly to open the discussion. The "simple" answer to this question may need to be scrutinized more carefully—as we shall do in the remainder of this chapter.

(2) Can you measure exactly how long the classroom is?

(3) If you are in doubt about your measurement, how doubtful are you? Could you be in error by one yard? by one foot? by one inch? by one-tenth of an inch? by one-hundredth of an inch? by one-millionth of an inch? Would your measurement be exact?

#### CLASS EXPERIMENT 1

(4) Have 10 people, working independently and in secret, guess the length of the classroom and write their guesses on a piece of paper. Give these 10 pieces of paper to a trustworthy person. We'll work with these numbers in the next few questions.

(5) How much doubt do you feel about these 10 guesses? Could they be in error by as much as 10 feet? by as much as one yard? by as much as one foot? by as much as one inch?

(6) Let's write all 10 guesses on the chalkboard, converting to the same unit in each case (probably the foot, and its decimal parts, is the best unit to use).

(7) Can you find the average of these 10 numbers? What is it?

(8) How much doubt do you feel about this average? Could it be in error by as much as 10 feet? by as much as one yard? by as much as one foot? by as much as one inch?

(9) We want to see how well these 10 people agreed with one another. (This is why we wanted to work independently, and to write their guesses in secret!) Mathematicians have thought of many different ways of comparing how well different measurements (or guesses) agree.

One method is to compute the range of the guesses.

For instance, suppose that the guesses were 31 ft, 30 ft, 33 ft, 27 ft, 32 ft, 32 ft, 29 ft, 30 ft, 34 ft, and 28 ft. Then the range (in the sense of statistics) of the 10 guesses would be 7 ft.

Can you see how to find the range of any number of guesses?

Do you think that range is a good measure of error?

(2) By rewording question 1, we may be able to get some children to wonder a bit.

(3) Presumably the children will feel their error would be less than one yard, and probably less than one-half yard, so that their measurement to the nearest yard would be "exact"—but only to the nearest yard. Not absolutely exact. They may also feel that their measurement to the nearest foot would be exact. When one looks for a measurement to the nearest inch, there may be more grounds for hesitation. And, for the smaller fractions of an inch, there is little doubt but that errors of this order will inevitably appear.

Evidently, one is unlikely to find the absolutely exact "right" answer! Since *no one* is likely to be able to find this, one can ask—on the philosophical level—whether any absolutely exact answer even exists! Surely, it is not known, in any event. Nobody knows the "right answers"!

(4) If you are in doubt about how Class Experiment 1 might be handled, we suggest you view the film entitled "Average and Variance."

(5) This is a matter for class discussion.

(6) Again, compare the film "Average and Variance." The use of decimal fractions of a foot is usually convenient.

(7) As usual, you find the average of the 10 numbers by adding them all up, and then dividing by 10.

(8) Again, a matter for class discussion.

(9) To find the range, look over the 10 guesses. See which one is largest (we'll call it  $L$ ). See which one is smallest (we'll call it  $s$ ). Subtract the smallest from the largest:

$$L - s.$$

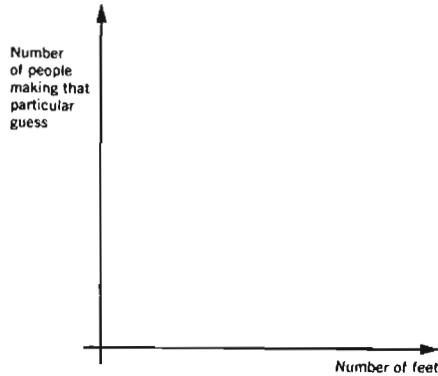
This answer is the range (which we might call  $R$ ):

$$L - s = R.$$

What number this will be, for your class, we cannot, of course, predict in advance. Evidently, the smaller  $R$  is, the greater agreement (roughly speaking) you have; the larger  $R$  is, the more disagreement you have—at least, to the extent that  $R$  is a "reasonable" measure of what you mean by "agreement."

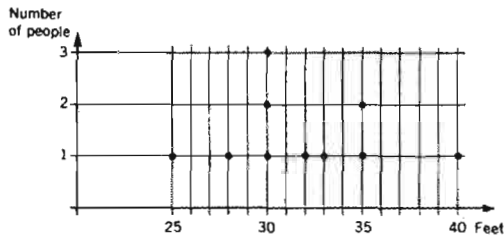
[page 53]

(10) A second method is to plot our points on a graph, like this:



Suppose, for example, the guesses were: 30 ft, 33 ft, 35 ft, 30 ft, 25 ft, 32 ft, 30 ft, 28 ft, 35 ft, 40 ft.

For these 10 guesses, our graph might be made to look like this:



This gives a kind of visual picture that suggests how well the different guesses agreed.

Why don't you make a graph using the 10 guesses from your class? How well did the people agree?

(11) Another method is the method of average absolute deviation from the average. (The name makes this method sound much harder than it really is. After a while this name will make sense to you, if you think about it.) We can illustrate this method, using sample data. If the guesses were 30, 33, 35, 28, 25, 32, 30, 28, 35, and 40, then we can find the average like this:

$$\begin{array}{r}
 30 \\
 33 \\
 35 \\
 28 \\
 25 \\
 32 \\
 30 \\
 28 \\
 35 \\
 \hline
 40 \\
 10 \overline{)316}
 \end{array}$$

31.6 is the average.

(10) This will depend upon the 10 numbers that your class has to work with.

(11) This, too, will depend upon your class.

Notice that it is not the *deviation from average* which we use, but rather *the absolute value of the deviation from average*. This point is not stressed in the student book—you may wish to emphasize it more yourself.

Now, 30 (the first guess) deviates from this average by this amount:

$$31.6 - 30 = 1.6.$$

so 1.6 is the deviation of the first guess from the average.

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The next guess, 33, deviates from the average by this much:

$$33 - 31.6 = 1.4.$$

Similarly, here are the deviations from the average for the other guesses:

$$\begin{aligned} 35 - 31.6 &= 3.4 \\ 31.6 - 28 &= 3.6 \\ 31.6 - 25 &= 6.6 \\ 32 - 31.6 &= 0.4 \\ 31.6 - 30 &= 1.6 \\ 31.6 - 28 &= 3.6 \\ 35 - 31.6 &= 3.4 \\ 40 - 31.6 &= 8.4 \end{aligned}$$

Consequently, the deviations (or deviations from the average) are:

$$1.6, 1.4, 3.4, 3.6, 6.6, 0.4, 1.6, 3.6, 3.4, 8.4.$$

What shall we do with these 10 numbers? The answer is that we will average them!

$$\begin{array}{r} 1.6 \\ 1.4 \\ 3.4 \\ 3.6 \\ 6.6 \\ 0.4 \\ 1.6 \\ 3.6 \\ 3.4 \\ \hline 8.4 \\ 34.0 \end{array} \qquad \begin{array}{r} 3.4 \\ \hline 10 \overline{)34.0} \end{array}$$

So, the average of the deviations from the average is 3.4, using our sample data. Can you compute the average absolute deviation from the average using the 10 guesses made in your class? For your 10 guesses, was the average absolute deviation from the average greater or less than that of our sample data? Which 10 guesses are more in agreement, yours or the 10 guesses in the sample data?

(12) Why do you suppose we call this the average absolute deviation from the average? Do you know what we mean by absolute value?

Notice that at this point we exercise some cunning in deciding the order of subtraction. Our procedure is to choose the order of subtraction so that the answer will be positive. This amounts to "taking the absolute value" of the deviation—hence the name "absolute deviation."

(12) Here we call attention to the point mentioned in our answer to question 11 above.

## CLASS EXPERIMENT 2

(13) Have 10 people measure the length of the room with 6-inch rulers. As before, the 10 people must work independently and in secret, and each must write his answer before seeing what any of the others have done. Give these 10 pieces of paper to a trustworthy person, who will keep them. We want to be able to work with these 10 numbers, and to refer back to them whenever we need to.

(14) How well do these 10 people agree? Could one of them be in error by as much as 10 feet? by as much as one yard? by one foot? by one inch? by one-tenth of one inch? by one-hundredth of one inch?

(15) Compute the average of these 10 numbers. Do you think the average could be in error by as much as one foot? by how much?

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(16) Compute the range of these 10 numbers. Did the "6-inch-ruler" measurements agree more, or less, than the guesses from Experiment 1?

(17) Use the method of graphs. Do the "6-inch-ruler" measurements seem to show more agreement, or less, than the guesses did?

(18) Use the method of average absolute deviation from the average. Do the 6-inch-ruler measurements show more agreement, or less, than the guesses did?

## CLASS EXPERIMENT 3

(19) Have 10 people measure the length of the room, using yardsticks or meter sticks of good quality. How well do the 10 people agree?

## CLASS EXPERIMENT 4

(20) Have 10 teams of people measure the length of the room, using high-quality tape measures. How well do the 10 teams agree?

(21) How would you find the exact length of the room?

(13) Compare, if you wish, the film "Average and Variance."

(14) A matter for class discussion.

(15) Again, add the 10 numbers and divide by 10, in the usual fashion.

(16) This will depend upon your class. The usual expectation—but be prepared for surprises!—would be that the range is smaller for the ruler measurements and larger for the guesses.

(17) See answer to question 16.

(18) See answer to question 16.

(19) This is shown in the film "Average and Variance."

(20) This, also, is shown in the film "Average and Variance."

(21) Evidently, you cannot!\*

\*Writers in general have been gaining a deeper understanding of the nature of science. As a result, one finds quite a few perceptive references to science. Consider, for example, the following:

... Finally Dundee turned around and faced them.  
 "Facts are facts," he said harshly.  
 Brager shook his head. "Not in science," he declared. "A fact is a phenomenon observed or recorded by an imperfect instrument."

This surprisingly sophisticated and accurate bit of dialogue occurs in a detective story, *The Sound of Murder*, by Rex Stout (Pyramid Books, paper back).

There is much more you can do with this topic. You can apply it to other measurement situations, to numerical determination of the number  $\pi$ , and so forth. The methods of testing the "consistency," or "degree of agreement," in a collection of numbers can be applied to numbers obtained in many ways. How consistent is an individual's response in some specific physiological situation? You can compare consistency for the same individual on different days, or compare consistency from one individual to another. How long, for example, can you stand on one leg? How long can you balance a 12-inch ruler on end in the palm of your hand?

You can probably think of many situations where consistency of data might be of interest. How many cards can you draw, one at a time, from a shuffled deck, *before* you draw an ace? How many automobiles pass by the school in ten minutes? How much variation is there—is it always the *same* number of cars in *any* ten-minute period?



## Part Four ■ Identities, Functions, and Derivations

chapter 18 / Page 56 of Student Discussion Guide

### IDENTITIES

By an *identity* we mean an open sentence that becomes true whenever a "legal" numerical substitution for each variable is made. For example,

$$\begin{aligned}\square + 0 &= \square \\ \square \times 1 &= \square \\ \square \times 0 &= 0 \\ 3 + \square &= \square + 3 \\ (\square \times \triangle) \times 0 &= 0 \\ \square \times \triangle &= \square \times \triangle \\ \square \times \triangle &= \triangle \times \square\end{aligned}$$

are all examples of *identities*. If you are unfamiliar with the subject of identities, merely be patient. We shall explore this subject in some detail in the next few pages.

The subject of identities is treated extensively in *Discovery*; its inclusion here can serve as a review, or else it can serve to make *Explorations* independent of any prior use of *Discovery*. Neither you nor your students are expected to be familiar with the topic of identities; we shall begin at the beginning and go on from there.



#### CHAPTER 18

#### Identities

[page 56]

Which are true? Which are false? Which are open?

- (1)  $29 + 51 = 70$
- (2)  $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$

### ANSWERS AND COMMENTS

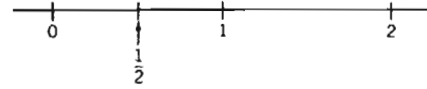
Questions 1 through 6 are intended partly to review the notions of *true*, *false*, and *open*.

- (1) **False**
- (2) **False**

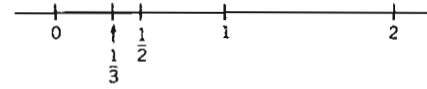
This question can play a diagnostic role; even students who have not yet learned any algorithms for adding fractions should recognize that this statement is false, provided they have some reasonable intuitive idea of what  $\frac{1}{2}$  means, what  $\frac{1}{3}$  means, and what  $\frac{1}{5}$  means. They might, for example, realize that  $\frac{1}{2}$  is a "good-

sized share," but that  $\frac{1}{3}$  is a "rather small share"; hence,  $\frac{1}{2}$  plus  $\frac{1}{3}$  should surely turn out to be more than  $\frac{1}{2}$ .

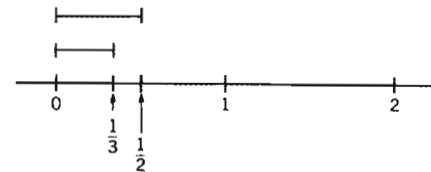
Alternatively, they can think in terms of the *number line* on which  $\frac{1}{2}$  would be marked here



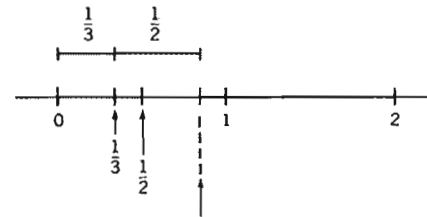
and  $\frac{1}{3}$  would be marked here



so that, if we add these two lengths together



we should get something closer to  $\frac{4}{3}$



and certainly not  $\frac{1}{3}$ .

(3)  $2 \times 3\frac{1}{2} = 7$

(4)  $5 + \square = 6$

(5)  $12 + \square = 12$

(6)  $\frac{1}{10} + \frac{1}{10} = 3\frac{1}{2}$

Can you find the truth set for each open sentence? -

(7)  $\cdot 8 + \square = \cdot 9$

(8)  $\cdot 8 + \square = \cdot 7$

(9)  $\cdot 8 + \square = 0$

(3) **True**

This question, also, can play a diagnostic role.

(4) **Open**

(5) **Open**

(6) **False**

(7)  $\{+1\}$

(8)  $\{-1\}$

(9)  $\{-8\}$



(10)  $(\square - 2) \times (\square - 3) = 0$

(11)  $\square \times \square = 16$

(12)  $\square \times \square = 169$

(13) Can you make up an open sentence that will become true for every legal substitution?

(10)  $\{+2, +3\}$ , or  $\{2, 3\}$ .

(11)  $\{+4, -4\}$

(12)  $\{+13, -13\}$

At this point, the main topic of this lesson begins.

(13) There are many possibilities. (It is important to remember that we must obey the rule for substituting, and put the same number in every  $\square$ , etc.) Here are a few that students often make up at the outset:

$$\begin{aligned} \square \times 0 &= 0 \\ 0 \times \square &= 0 \\ \square + 0 &= \square \\ \square \times 1 &= \square \\ \square \times 2 &= \square \times 2 \\ \square \times 2 &= 2 \times \square \\ \square + \square &= \square + \square \\ \square + \square &= \square \times 2 \\ \square - \square &= 0 \\ \square + \square + \square &= 3 \times \square \\ \square + 7 &= 7 + \square \\ \square + 3 &= (\square + 2) + 1 \\ &\vdots \end{aligned}$$

You may wish to view the film "Second Lesson."

(14) Jerry says this open sentence will become true for every substitution:

$$\square \times 0 = 0.$$

Do you agree?

(15) Do you know what we mean by an identity?

(14) Jerry is correct.

(15) Notice that in questions 13 and 14 we began *thinking about the concept of identity*. Now, we introduce for the first time the word *identity*. This question may be rhetorical, and may very likely have to be answered by the teacher. Nonetheless, we believe we get better attention from the class by asking this as a question, instead of merely giving it as a statement. Here are several possible "suitable" answers, which can be adjusted to the sophistication level of your class:

"An identity is an open sentence that becomes true for every numerical replacement of the variables."

**"An identity is an open sentence that becomes true for every legal substitution."**

**"An identity is an open sentence where 'every number works'; the truth set is the set of all numbers."**

As a matter of fact, the various descriptions of what we shall mean by an "identity" are not completely equivalent. Consider, for example, the open sentence

$$\frac{\square}{\square} = 1.$$

Recalling that division by zero is never a legal operation, we see that for this open sentence every *legal* numerical replacement produces a true statement, but it is not correct to say that "every numerical replacement" does so. Other differences occur when we "put  $A + B$  into the  $\square$ "—i.e., when we use what is sometimes called an "open name" as a replacement for a variable. These differences are not, however, serious difficulties so long as the basic concept of "identity" is understood; the students recognize the fact that our descriptions are merely attempts—more or less imperfect—to put into words the *idea* of what we mean by an "identity"; the various exceptional cases are treated honestly, if the need to do so arises.

(16) Sarah says that mathematicians use the symbol " $\forall_x$ " to mean "for all  $x$ ," and that they would write Jerry's idea this way:

$$\forall_x \quad x \cdot 0 = 0.$$

(16) **Sarah is correct. The symbol  $\forall_x$  means "for all  $x$ " or "you may make any legal replacement for the variable  $x$ , and the following open sentence will become true . . .," or something of this sort. For example,**

$$\forall_x \quad x \cdot 0 = 0$$

**might be read as "for all  $x$ ,  $x$  times zero equals zero," or it might be read as "you may make any legal replacement for the variable  $x$ , and the open sentence  $x \cdot 0 = 0$  will become true."**

It is probably best to allow some small variation in our usage of the symbol  $\forall_x$ .

Incidentally, the symbol  $\forall_x$  is taken from the subject of mathematical logic, and is becoming increasingly important in "modern" approaches to mathematics. See Appendix A, Suppes (98). This symbol is referred to, by logicians, as a "quantifier." Present Madison Project materials make extensive use of quantifiers at the ninth-grade level. For the present, we merely introduce the symbol in a relatively casual way, in order that it may be familiar when it is needed in subsequent work. We would leave it out entirely with very young children.

(17) Can you make up any more identities?

**IMPORTANT:** For our future work we will need a long list of identities. The best way to get such a list will be for you to maintain a "cumulative" list of identities as you make them up. Keep this list in a safe place where we can refer to it whenever we may need to.

(17) **There are, of course, a tremendous number of possibilities.**

The Madison Project film entitled "Second Lesson" shows a group of children, from grades 3 through 7, learning about identities for the first time. You may want to view it when you are teaching identities to your own students.

Be sure the students preserve their lists of identities carefully. You may prefer to have individual students accumulate their own lists, or you may prefer to have a single "official" list for the entire class.

## MAKING UP SOME “BIG” IDENTITIES BY PUTTING TOGETHER “LITTLE” ONES

If someone shows you an open sentence and asks you if it is an identity, you may have—at this stage in our work—no very satisfactory method for deciding. For example, is

$$\square \times (\square + 1) = (\square \times \square) + \square$$

an identity? One method which students often use is to look at it to see if it “looks”—I suspect they mean “feels”—like it ought to be an identity. This method, in the hands of our students, is *very unreliable*. For the example above, they usually argue that it is *not* an identity “because it has two boxes on one side, and three boxes on the other.” This argument is irrelevant and, in fact, quite wrong in this instance.

A far better method is to go back to the definition: if *every* “legal” numerical replacement of the variable produces a *true* statement, then the open sentence is an identity (as the children say, “every number works”). If you can find a single legal replacement that yields a *false* statement, then you have the matter settled definitely and finally. In such a case, the open sentence is not an identity, because you have shown that it is not true that “every number works.” You just found one that didn’t!

Ah . . . but what if every substitution that you try yields a *true* statement? What then? You do not know that “every number works,” for you have only tried a few numbers. There are infinitely many different numbers, and you can never complete the job of trying them all. In this case, your result is only tentative; as far as you know, the open sentence *seems* to be an identity. You cannot, however, be sure. Perhaps the very next number you try will yield a false statement. How can you be *sure* that it won’t? You can’t.

There are probably two other methods that children use, in trying to decide whether an open sentence is an identity. One is to see if there seems to be any reason why it should be; for example,

$$\square + 0 = \square.$$

If you “don’t add any more,” you still have the same number you started out with. The other method is to see if you can retrace the steps that the other person followed in making up the identity. For example, in

$$\square + 12 = 1 + \frac{2}{3} + \square + \frac{2}{3} + 6 + 2,$$

it seems clear that the man who made this up started with

$$\square + 12 = \quad ,$$

and, for that weird right-hand side, he broke the 12 up into bits and pieces and sort of scattered them around. But, if you put them

back together, the bits and pieces *do* add up to 12. Hence, you might guess that this *is* an identity.

We shall not try to describe all this at the verbal level. It would get too complicated. However, we *do* want to start building some subverbal comprehension of identities in our students. For this purpose, the following game seems to do nicely.

There are two teams, Team A and Team B. Team A makes up an open sentence and shows it to Team B (probably by writing it on the chalkboard). Now, Team B must try to decide whether or not that open sentence is an identity. The teacher adjudicates, and announces whether Team B is right or wrong (of course, the students can argue, if they have any real evidence and can prove that the teacher made a mistake). If Team B guessed correctly, they get 10 points. If Team B guessed incorrectly, then Team A gets 10 points.

Now the roles are reversed, and Team B makes up an open sentence for Team A to classify either as an identity or as not an identity. The game continues until one team has, say, 200 points (or you may use some other agreed-upon method for determining when the game is over).

This game provides valuable experience that will be useful in Chapter 19 and in other following chapters. [If you object to the competitive nature of team games, you can modify the rules. Eliminate the teams, and simply have students pose problems for the rest of the class to discuss. For an interesting view of such matters, consult Appendix A, Henry (42).]

In the work thusfar on identities, we have made up open sentences which we believed to be identities, and we have looked at open sentences made up by someone else and tried to decide whether or not they were identities.

All of this work was done in a preliminary, "intuitive," nonverbalized fashion. Having *done* some such work, we can now try to *describe* it. As we make our description more fully explicit, and more minutely detailed, we develop the idea of a derivation, the idea of axioms versus theorems, and the explicit set of rules—which we call a "logic"—for making up derivations.

This follows our familiar pattern of

- (i) action,
- (ii) intuitive "description" of the action,
- (iii) explicit description of the action.

We use this sequence again and again; in this we are probably influenced by the work of Professor Jerome Bruner, of Harvard University, who describes "inductive" or "experiential" learning as consisting of the sequence of

- (i) performing an action,
- (ii) building mental imagery to represent cognitively a part of what the action did in reality,
- (iii) building explicit notation to name either the action or its cognitive image.

Now, the explicit description of how we make up "big" identities out of "little" ones—or *new* identities out of *old* ones—will, at this stage of our work, depend primarily upon two processes: our old friends UV and PN.



## CHAPTER 19

## Making Up Some “Big” Identities by Putting Together “Little” Ones

## ANSWERS AND COMMENTS

[page 57]

In making up identities, you probably had a method – whether conscious or unconscious. For example, you may have started an identity something like this

$$[(\square + 3) \times \triangle] \times (\triangle + \frac{1}{2})$$

and then thought to yourself, “Ah! If I now multiply by zero, the result will be zero.” Consequently, you write

$$\{[(\square + 3) \times \triangle] \times (\triangle + \frac{1}{2})\} \times 0 = 0.$$

In a simpler case, you may have begun

$$\square \times \square$$

and then thought, “Aha! If I now add zero, the sum will be unaffected by the addition of zero,” and so you wrote

$$(\square \times \square) + 0 = \square \times \square.$$

As a third line of reasoning, you may have begun with

$$(\square + \frac{1}{2}) + (\square + 3)$$

and said to yourself, “Why, all you have to do is to put exactly the same thing on the other side of the equals sign, and surely that will give you an identity!” Consequently, you wrote:

$$(\square + \frac{1}{2}) + (\square + 3) = (\square + \frac{1}{2}) + (\square + 3).$$

In this chapter we want to investigate these methods for making “fancy” identities out of other, simpler ones.

Probably the best way to carry on our investigation is to look at a few examples.

**Example 1**

Sometimes you use UV (use of variables).

You might start with a simple identity, like

$$\square + \triangle = \triangle + \square.$$

and then you use UV to get a more complicated identity. Suppose, for example, we do this:

$$UV: A + B \rightarrow \square$$

$$\frac{1}{2} + \frac{3}{4} \rightarrow \triangle$$

The result will be [page 58]

$$\boxed{A + B} + \triangleleft_{\frac{1}{2} + \frac{2}{3}} = \triangleleft_{\frac{1}{2} + \frac{2}{3}} + \boxed{A + B}$$

which we would ordinarily write as

$$(A + B) + \left(\frac{1}{2} + \frac{2}{3}\right) = \left(\frac{1}{2} + \frac{2}{3}\right) + (A + B).$$

We have made up this "more complicated" identity by using UV.

If you prefer  $\square$ 's and  $\triangle$ 's, instead of  $A$ 's and  $B$ 's, to indicate your variables, you can use UV again, like this:

$$\text{UV: } \square \rightarrow A$$

$$\triangle \rightarrow B$$

$$(\square + \triangle) + \left(\frac{1}{2} + \frac{2}{3}\right) = \left(\frac{1}{2} + \frac{2}{3}\right) + (\square + \triangle)$$

### Example 2

Sometimes you use PN (principle of names).

We might, for instance, start with the identity we just got. We could make it still more complicated if we want to. For example,

$$(i) \quad \square + \triangle = \triangle + \square$$

and we just got the identity

$$(ii) \quad (\square + \triangle) + \left(\frac{1}{2} + \frac{2}{3}\right) = \left(\frac{1}{2} + \frac{2}{3}\right) + (\square + \triangle)$$

We could use PN—the method of "erasing" one name and putting in its place another name for the same thing—to get:

$$(iii) \quad (\square + \triangle) + \left(\frac{1}{2} + \frac{2}{3}\right) = \left(\frac{1}{2} + \frac{2}{3}\right) + (\triangle + \square)$$

PN from line (ii),  
using line (i).

Q.E.D.\*

(Remember, the heavy underlining shows which name was "erased" and replaced by another name for the same thing.)

[page 59]

(1) Try to make up some more identities to add to your cumulative list. (Remember to keep your list carefully; we shall need it later.)

(2) Start with the identity

$$\nabla + 0 = \nabla,$$

and use UV like this:

$$\text{UV: } A + B + \frac{1}{2} \rightarrow \nabla.$$

(1) This will depend upon your class.

$$(2) (i) \quad \nabla + 0 = \nabla$$

$$(ii) \quad (A + B + \frac{1}{2}) + 0 = A + B + \frac{1}{2}$$

$$\text{UV: } A + B + \frac{1}{2} \rightarrow \nabla$$

in line (i).

\*Q.E.D. stands for the Latin term *quod erat demonstrandum*. It means we have now proved what we were asked to prove.

What result do you get? Now replace the  $A$ 's and  $B$ 's by  $\square$ 's and  $\triangle$ 's. What is your final result?

(3) Start with the identity

$$(\square + \triangle) + [\square \times (\triangle + \nabla)] = (\square + \triangle) + [\square \times (\triangle + \nabla)].$$

Now use PN, and use these identities:

$$\square + \triangle = \triangle + \square$$

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla).$$

Can you get a more complicated identity as a result? What result did you get? Can you write out each step carefully?

$$(iii) (\square + \triangle + \frac{1}{2}) + 0 = \square + \triangle + \frac{1}{2}$$

$$UV: \square \rightarrow A$$

$$\triangle \rightarrow B^*$$

in line (ii).

(3) There are several possibilities. Here is one:

$$(i) (\square + \triangle) + [\square \times (\triangle + \nabla)] \\ = \underline{(\square + \triangle)} + [\square \times (\triangle + \nabla)]$$

$$(ii) \square + \triangle = \triangle + \square$$

$$(iii) (\square + \triangle) + [\square \times (\triangle + \nabla)] \\ = \underline{(\triangle + \square)} + [\square \times (\triangle + \nabla)]$$

PN, from line (i), using line (ii).

$$(iv) (\square + \triangle) + [\square \times (\triangle + \nabla)] \\ = (\triangle + \square) + [\square \times (\triangle + \nabla)]$$

Here we have repeated line (iii), in order to make sure that the *underlining* for our *first* use of PN (using line ii) will not get confused with the underlining for our *second* use of PN (where we shall be obtaining line vi by using line v).

$$(v) \square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

$$(vi) (\square + \triangle) + [(\square \times \triangle) + (\square \times \nabla)] \\ = (\triangle + \square) + [\square \times (\triangle + \nabla)]$$

PN, from line (iv), using line (v).

\*In this identity we have indicated the correct placement of parentheses. That is to say, we have not made a decision between  $(\square + \triangle) + \frac{1}{2}$  vs.  $\square + (\triangle + \frac{1}{2})$ . There is a very good reason why we have not done so: extensive trials with children have convinced us that this question of "correct placement of parentheses in addition problems" can profitably go through three stages. First, we "just add," and ignore parentheses altogether. This is—at first—a natural procedure for children. At the second stage, armed with experiences where parentheses play a decisive role (as in the case of the "distributive law"), we raise the question of "where the parentheses really ought to go?" Finally, in the third stage, we "discover" (or recognize) the *associative law for addition*, and thereafter can deal with the matter carefully whenever it seems appropriate to do so. Perhaps we should really recognize a fourth stage: once we know that we can deal with ALA carefully, we allow ourselves to become careless in situations where meticulous care seems unnecessary. But all of this is getting ahead of our story. We are now only at Stage 1, where we are ignoring the question for the time being.

## SHORTENING LISTS: “AXIOMS” AND “THEOREMS”

There are three mathematical ideas in this chapter.

(i) Sometimes introducing another variable permits you to replace a long list of identities by a single identity, without losing any information. For example, if we have the list

$$\begin{aligned}\square + 1 &= 1 + \square \\ \square + 2 &= 2 + \square \\ \square + 3 &= 3 + \square \\ \square + 4 &= 4 + \square \\ &\vdots \\ &\vdots\end{aligned}$$

we can replace this entire list by *one single identity*, namely

$$\square + \triangle = \triangle + \square.$$

Notice that, in doing this, we introduced another variable (which, since the symbol had not already been used in this open sentence, we chose to write as  $\triangle$ ). Notice also that, in order to recover any identity on the original list, we need only UV. For example,

$$\text{UV: } 3 \rightarrow \triangle$$

gives us back the third identity on the original list.

(ii) In Chapter 19 we actually began writing out derivations (see Chapter 19, problem 3). We can now use such derivations as *another* means for shortening certain lists.

(iii) Finally, suppose that we made up the longest list that we could, consisting of identities and true statements in arithmetic and algebra. Suppose we then “shortened” this list by the two methods just mentioned. We would finally end up with a list which could not be shortened any further without actually losing some information.

The statements or identities on this final list are what mathematicians call *axioms*. The statements which were eliminated from the list during our process of “shortening” it are called *theorems*.

As usual, this will become clearer as we actually get into our work. Compare, again, Professor Bruner’s sequence for “experiential learning”: action, then imagery, then notation. What we need now is some action, so we shall turn immediately to our problems for this chapter.





## CHAPTER 20

## Shortening Lists: "Axioms" and "Theorems"

[page 60]

(1) Jeanne has this list of identities:

$$(\square \times \triangle) + 3 = (\triangle \times \square) + 3$$

$$(\square \times \triangle) + 4 = (\triangle \times \square) + 4$$

$$(\square \times \triangle) + 5 = (\triangle \times \square) + 5$$

$$(\square \times \triangle) + 6 = (\triangle \times \square) + 6$$

⋮

What do you suppose the three dots at the bottom mean? Can you make up any more identities that "look like" those on Jeanne's list—that is to say, that have this same pattern?

(2) Albert says he can write one single identity to represent Jeanne's entire list. Do you think he can? How?

(3) Suppose you had Albert's single identity. Could you get the identity

$$(\square \times \triangle) + 5 = (\triangle \times \square) + 5$$

from Albert's by using UV? How?

(4) Anne has this list of identities:

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

$$A + (B \times C) = (C \times B) + A$$

Could you shorten Anne's list, without really losing anything?

## ANSWERS AND COMMENTS

(1) The three dots indicate that the list can always be extended further. Obviously, you can make up lots more identities like this. One fifth-grade class recently pointed out that we really have here a list that can be continued indefinitely in either direction:

$$(\square \times \triangle) + -2 = (\triangle \times \square) + -2$$

$$(\square \times \triangle) + -1 = (\triangle \times \square) + -1$$

$$(\square \times \triangle) + 0 = (\triangle \times \square) + 0$$

$$(\square \times \triangle) + 1 = (\triangle \times \square) + 1$$

$$(\square \times \triangle) + 2 = (\triangle \times \square) + 2$$

$$(\square \times \triangle) + 3 = (\triangle \times \square) + 3$$

⋮

(2) Here it is:  $(\square \times \triangle) + \nabla = (\triangle \times \square) + \nabla$

Notice that we have introduced a new variable. (We chose to write it as  $\nabla$ , but we might have written it with any "variable" symbol except  $\square$  or  $\triangle$ . We could not use  $\square$  or  $\triangle$ , since each of these is involved in the open sentence already. We might, however, have written it as  $\square$ ,  $\square$ ,  $\square$ , or whatever. We do not use circles, because they can easily be confused with zeros.)

(3) Take

$$(\square \times \triangle) + \nabla = (\triangle \times \square) + \nabla$$

and use UV

$$UV: 5 \rightarrow \nabla$$

to get

$$(\square \times \triangle) + 5 = (\triangle \times \square) + 5.$$

(4) You could delete the identity  $A + (B \times C) = (C \times B) + A$ , so that Anne's "shortened" list would become:

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

Thus, the two identities above might be taken as *axioms*, and the identity

$$A + (B \times C) = (C \times B) + A$$

would be a *theorem*. Now, we need to prove that, when we deleted

$$A + (B \times C) = (C \times B) + A,$$

we *didn't really lose anything*. We prove this by showing how to reconstruct

$$A + (B \times C) = (C \times B) + A$$

from the other two identities, using only UV and PN. For convenience of notation, let's number the axioms:

Axiom 1.  $\square + \triangle = \triangle + \square$

Axiom 2.  $\square \times \triangle = \triangle \times \square$

Here we go:

Step	Reason
(i) $\square + \triangle = \triangle + \square$	Axiom 1.
(ii) $A + (B \times C) = \underline{(B \times C)} + A$	UV: $A \rightarrow \square$ $B \times C \rightarrow \triangle$ in line (i).
(iii) $\square \times \triangle = \triangle \times \square$	Axiom 2.
(iv) $B \times C = C \times B$	UV: $B \rightarrow \square$ $C \rightarrow \triangle$ in line (iii).
(v) $A + (B \times C) = \underline{(C \times B)} + A$	PN from line (ii), using line (iv).

(5) Marjory says she could shorten Anne's list to this:

$$\begin{aligned} \square + \triangle &= \triangle + \square \\ \square \times \triangle &= \triangle \times \square \end{aligned}$$

Marjory says that once you know these two, you can always make up

$$A + (B \times C) = (C \times B) + A,$$

by using UV and PN. What do you think?

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(6) Take your list of identities and shorten it as much as possible, without really losing anything. What does your final list look like?

(5) Marjory is right. (See the answer to question 4.)

(6) This will depend upon your class. If your long list used only addition and multiplication and if it avoids fractions and negative numbers, your final "shortened" list may well look like this (where we give the standard name for each statement, as well):

$\square = \square$	Reflexive Property of Equality (RPE)
$\square + 0 = \square$	Addition Law for Zero (ALZ)
$\square \times 0 = 0$	Multiplication Law for Zero (MLZ)
$\square \times 1 = \square$	Law for One (L1)
$\square + \triangle = \triangle + \square$	Commutative Law for Addition (CLA)
$\square \times \triangle = \triangle \times \square$	Commutative Law for Multiplication (CLM)
$\left. \begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ 4 + 1 = 5 \\ \vdots \end{array} \right\}$	Definition of the Numerals* 2, 3, 4, ... (Def. Num.)

There are other identities that might also appear on your "shortest list," including these:

$\square \times (\triangle + \nabla)$	
$= (\square \times \triangle) + (\square \times \nabla)$	Distributive Law (DL)
$\square + (\triangle + \nabla)$	
$= (\square + \triangle) + \nabla$	Associative Law for Addition (ALA)
$\square \times (\triangle \times \nabla)$	
$= (\square \times \triangle) \times \nabla$	Associative Law for Multiplication (ALM)

The "shortest list" of identities that you end up with will, of course, depend upon the "long" list of identities that you started with, and these long lists will be different for different classes.

(7) Do you know what mathematicians mean by the word **axiom**?

(7) An **axiom** is one of the "basic" or "building-block" statements (or identities) from which all the rest of our mathematical system can be derived.

(8) Do you know what mathematicians mean by the word **theorem**?

(8) A **theorem** is a true statement, or an identity, which is not an axiom. (We try to keep our list of axioms as short as possible.)

\*In some Madison Project materials these statements are referred to collectively as "Changing Names" (CN). This designation was given them by a class of fifth-graders.

## HOW SHALL WE WRITE DERIVATIONS?

This chapter follows easily from Chapter 20. There is, however, one decision to make. There are two possible ways to derive

$$A + (B \times C) = (C \times B) + A.$$

One method starts from

$$\square = \square$$

(and is given on page 62 of the *Student Discussion Guide*), whereas the other starts from

$$\square + \triangle = \triangle + \square,$$

(and is given on page 180 of the teachers' text).

We believe that the derivation starting from  $\square = \square$  is easier to get started, but it is clearly longer. The derivation starting from  $\square + \triangle = \triangle + \square$  is shorter, but may be tricky to get started on.

Since students often say they "don't know where to begin," the longer derivation beginning with  $\square = \square$  may, in fact, actually be easier.

Use whichever you and your students prefer (or, if you prefer, use both and compare them).



### CHAPTER 21

## How Shall We Write Derivations?

[page 62]

Debbie claimed that she could use

Axiom 1:  $\square = \square$

Axiom 2:  $\square + \triangle = \triangle + \square$

Axiom 3:  $\square \times \triangle = \triangle \times \square$

together with UV and PN and end up with

$$A + (B \times C) = (C \times B) + A.$$

George challenged Debbie to prove it, and so Debbie wrote this:

Statement	Reason
(i) $\square = \square$	Axiom 1.

## ANSWERS AND COMMENTS

(ii)  $A + (B \times C) = A + \underline{(B \times C)}$

UV:  $A + (B \times C) \rightarrow \square$ ,

in line (i).

(iii)  $\square \times \triangle = \triangle \times \square$  Axiom 3.

(iv)  $B \times C = C \times B$  UV:  $B \rightarrow \square$

$C \rightarrow \triangle$

in line (iii).

(v)  $A + (B \times C) = A + \underline{(C \times B)}$

PN from line (ii),  
using line (iv).

(vi)  $\square + \triangle = \triangle + \square$  Axiom 2.

(vii)  $A + (C \times B) = (C \times B) + A$

UV:  $A \rightarrow \square$

$C \times B \rightarrow \triangle$

in line (vi).

(viii)  $A + (B \times C) = \underline{A + (C \times B)}$

Repeat of line (v), in  
order to avoid confusing  
the underlining for PN.

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(ix)  $A + (B \times C) = \underline{(C \times B) + A}$

PN from line (viii),  
using line (vii).

Q. E. D.

(1) Who won the argument, George or Debbie?

(2) Andy says he can make an even shorter derivation that will be just as good as Debbie's. Can you? Do you think Andy can?

(3) Study Debbie's derivation very carefully, and then try to make your own derivation that uses the axioms

$$\square = \square$$

$$\square + \triangle = \triangle + \square$$

$$\square \times \triangle = \triangle \times \square$$

and ends up with the theorem

$$(A + B) \times C = C \times (B + A).$$

Try to give reasons, the same way that Debbie did.

(1) Debbie won. Her proof is correct (and very nicely written!).

(2) Answers will vary.

(3) Here we go:

Step	Reason
(i) $\square = \square$	Axiom 1.
(ii) $(A + B) \times C$ $= \underline{(A + B)} \times C$	UV: $(A + B) \times C \rightarrow \square$ in line (i).
(iii) $\square + \triangle = \triangle + \square$	Axiom 2.
(iv) $A + B = B + A$	UV: $A \rightarrow \square$ $B \rightarrow \triangle$ in line (iii).

$$\begin{aligned} \text{(v)} \quad & (A + B) \times C \\ & = \underline{(B + A)} \times C \end{aligned}$$

PN, from line (ii), using line (iv).

$$\begin{aligned} \text{(vi)} \quad & (A + B) \times C \\ & = \underline{(B + A)} \times C \end{aligned}$$

Repeat of line (v), so that the underlining for our first use of PN will not get mixed up with our underlining for our second use of PN, which follows.

$$\text{(vii)} \quad \square \times \triangle = \triangle \times \square$$

Axiom 3.

$$\begin{aligned} \text{(viii)} \quad & (B + A) \times C \\ & = C \times (B + A) \end{aligned}$$

UV:  $B + A \rightarrow \square$

$C \rightarrow \triangle$   
in line (vii).

$$\begin{aligned} \text{(ix)} \quad & (A + B) \times C \\ & = \underline{C \times (B + A)} \end{aligned}$$

PN, from line (vi) using line (viii).

**Q.E.D.**

Notice that a shorter proof is possible. It may, however, be trickier to know how to start it. Can you see how to do it? (Hint: Try starting with  $\square + \triangle = \triangle + \square$ , then use UV as

$$\begin{aligned} \text{UV: } A + B & \rightarrow \square \\ C & \rightarrow \triangle.) \end{aligned}$$

I am sorely tempted to write it out here, myself, because I think these are fun—but I fear that a teacher who has not had the experience of discovering original derivations herself will not honestly believe that her students can discover derivations themselves.

Probably I should not even have given the hint. Oh well, here is a new problem you can work on *by yourself*—I won't interfere: Start with the axioms

$$\begin{aligned} \square & = \square \\ \square \times (\triangle + \nabla) & = (\square \times \triangle) + (\square \times \nabla) \\ \square \times \triangle & = \triangle \times \square \end{aligned}$$

and make up an original derivation for the theorem

$$A \times (B + C) = (B \times A) + (C \times A).$$

## SUBTRACTION AND DIVISION

In making up a mathematical system, one always has the problem of where to start. When one assumes a computational or "counting" point of view, addition is a "basic" process—it is new when one encounters it, and cannot be explained in terms of anything that has preceded it. However, multiplication, from this point of view, is *not* "new"—multiplication can be explained in terms of addition (that is, in terms of repeated addition,  $2 \times 7 = 7 + 7$ , and so on).

If, instead, one begins with *sets*, then the union and intersection of sets are basic ideas, and addition and multiplication are no longer basic; addition and multiplication can be explained in terms of set operations.

In the present volume, we take neither of the preceding points of view, but lean chiefly toward a third point of view—the one which has been current in recent decades among those mathematicians who study "modern algebra." This is an *axiomatic* approach. Using it, we take *addition* and *multiplication* to be our basic notions, and do not try to explain them in terms of anything prior. The operations of *subtraction* and *division*, from an axiomatic point of view, are then explained in terms of addition and multiplication.

We have all known this approach for some time; we use it whenever we use addition to "check" subtraction, or use multiplication to "check" division. If such "checking" is possible, then it must be possible for addition to tell us whether or not a subtraction has been performed correctly. But if addition can do this much for subtraction, then there must be nothing *logically* new about subtraction—and, indeed, there is not.

There are several ways to reduce subtraction to addition. We shall do it by means of *additive inverses*. If we start with a number, say,  $+3$ , we can add a number so that the result will be zero:

$$+3 + \underline{\quad} = 0.$$

Obviously, the number to add is  $-3$ . We shall call  $-3$  the *additive inverse* of  $+3$ .

To make matters clearer, let's consider a few more examples. What number would be the additive inverse of  $+7$ ? It would be the number we could add to  $+7$ , in order to get a sum of zero:

$$+7 + \underline{\quad} = 0.$$

Evidently this number is  $-7$ ; consequently, we say that  $-7$  is the additive inverse of  $+7$ .

What would happen if we sought the additive inverse of a negative number? As an example, what number would be the additive inverse of  $-2$ ? By the definition of additive inverse, it would be the number that we could add to  $-2$  to get a sum of zero:

$$-2 + \underline{\quad} = 0.$$

Evidently, it would be  $+2$ ; we could say that the additive inverse of  $-2$  is  $+2$ .

For practice, let's find the additive inverse of  $-\frac{1}{2}$ . That would be "the number we must add to  $-\frac{1}{2}$  to get a sum of zero":

$$-\frac{1}{2} + \underline{\quad} = 0.$$

Again, it is clear what the number must be; in this case it is  $\frac{1}{2}$ . We can express this by saying that the additive inverse of  $-\frac{1}{2}$  is  $\frac{1}{2}$ .

As a last example, and a particularly interesting one, let's try to find the additive inverse of zero. That is, we are seeking a number that we can add to zero, to get a final sum of zero:

$$0 + \underline{\quad} = 0.$$

Evidently, the desired number is 0:

$$0 + 0 = 0.$$

We can express this by saying that the additive inverse of zero is zero, or zero is its own additive inverse.

A word of caution! A few years ago, in an attempt to appeal to the intuition, some mathematicians introduced the word *opposite* as a synonym for *additive inverse*. Thus, using this word, we could say "the opposite of +3 is -3." This introduces a hazard we have mentioned elsewhere: the word *opposite* now has a *mathematical* meaning, and it also has an *everyday* meaning. *These meanings are not the same!* In mathematics, the word must be used *only* in its *mathematical* meaning; otherwise confusion will result. If we even let ourselves *think* its everyday meaning, this can confuse us. Consider the statement "the additive inverse of zero is zero." Translated into "opposite" language, this becomes "the opposite of zero is zero." *Mathematically*, what does this say? It says that if you start with zero and add zero, the resulting sum will be zero.

The *everyday* meaning of the word *opposite*, in this case, appears to result in a ridiculous statement. One should never confuse *everyday* meanings with *mathematical* meanings. They are different.

Now that we know what we mean by *additive inverse* (or, in its mathematical sense, *opposite*), we can introduce subtraction with no further effort. We shall now hereby officially agree that, whenever we say

$$A - B$$

what we really mean is

$$A + {}^{\circ}B,$$

where the notation " ${}^{\circ}B$ " denotes the additive inverse of  $B$ . That is to say, whenever we are asked to *subtract*  $B$ , what we shall do, instead, will be to *add* the **additive inverse** of  $B$ . This will dispose of all of the questions involving subtraction; once we have seen how this works out, we shall use a closely analogous procedure to dispose of division (in terms of what we shall call *multiplicative inverses*).

Our present procedure of explaining subtraction in terms of addition, and explaining division in terms of multiplication, has at least three important advantages.

(i) It gives us a precise language and precise criteria for settling all "doubtful" cases that may arise. For example, the difficulties involved in discussing "division by zero" can be handled *precisely* in this fashion, and do not require appeals to vague arguments.



(ii) Once we have handled subtraction, we can use *almost exactly* this same procedure to discuss division. Thus  $^+3 - ^+2$  involves virtually the same ideas as  $\frac{3}{7} \div \frac{2}{8}$ , although in traditional treatments this latter problem is usually much harder to understand.

(iii) What we are doing can easily be extended to other future mathematical systems, however abstract or pathological these future systems may be. The "traditional" approaches usually cannot be extended to modern abstract systems.

All of these matters will become clearer as we work through the questions in this chapter.



## CHAPTER 22

## Subtraction and Division

[page 64]

(1) Nancy says that we know lots of important identities involving **addition** and **multiplication**, but we do *not* have any for **subtraction** or **division**. What do you think?

(2) Tony says you can handle **subtraction** by turning it into **addition**. Do you know what Tony means?

(3) Do you know what mathematicians mean by the **additive inverse** or the **opposite** of a number?

(4) Can you find other names for these numbers?

(a)  $^{\circ}(-1)$

(b)  $^{\circ}(2\frac{1}{2})$

(c)  $^{\circ}(-3)$

(d)  $^{\circ}(\frac{1}{2})$

(e)  $^{\circ}(0)$

## ANSWERS AND COMMENTS

(1) Obviously this depends upon exactly what you have done in your class, but we would expect Nancy's statement to be correct in most cases.

(2) This refers to what we discussed in the introduction to this chapter. Your students, however, will probably *not* know at this point. The question is *not* intended to be answered at this stage. Rather, this question is intended to focus student attention on the problem on which we now want to work.

(3) The **additive inverse** of, say,  $^+1985$  is the number we must add to  $^+1985$  in order to get a sum of zero:

$$^+1985 + \underline{\hspace{2cm}} = 0.$$

*Opposite, in its mathematical sense, means exactly the same thing as additive inverse.*

(4) **Note:** the symbol used in this question (a small raised circle, to the upper left of a number) means "the opposite of" or "the additive inverse of." Thus,  $^{\circ}(+1)$  would be read as the opposite of positive one, or as the additive inverse of positive one.

(a)  $^{\circ}(+1) = -1$

(b)  $^{\circ}(2\frac{1}{2}) = -2\frac{1}{2}$

(c)  $^{\circ}(-3) = +3$

This would be read as: "the opposite of negative three is positive three," or as "the additive inverse of negative three is positive three."

(d)  $^{\circ}(\frac{1}{2}) = +\frac{1}{2}$

(e)  $^{\circ}(0) = 0$

- (f)  $^{\circ}(3)$   
 (g)  $^{\circ}(^{\circ}3)$   
 (h)  $^{\circ}[^{\circ}(^{\circ}3)]$   
 (i)  $^{\circ}[+2 + +3]$

(5) Jean says Cynthia used to have a "rainbow picture" to show what we mean by opposites. Do you know what Jean is talking about?

(6) Debbie says the official definition of the opposite of  $A$  is "the number that I can add to  $A$  so that the sum will be zero." What do you think?

(7) Which of these are identities?

- (a)  $\square + ^{\circ}\square = 0$   
 (b)  $\square + \square = 0$   
 (c)  $\square + \square = \square$

See the discussion at the beginning of this chapter. Notice that part e is one place where the "everyday" meaning of the word *opposite* could confuse you badly.

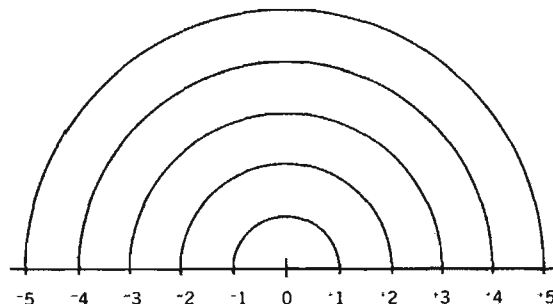
- (f)  $^{\circ}(3) = -3$   
 (g)  $^{\circ}(^{\circ}3) = ^{\circ}(-3)$   
 $= +3$   
 (h)  $^{\circ}[^{\circ}(^{\circ}3)] = ^{\circ}[^{\circ}(-3)]$   
 $= ^{\circ}[+3]$   
 $= -3$   
 (i)  $^{\circ}(+2 + +3) = ^{\circ}(+5) = -5$

Incidentally, you would have obtained this same result if you had said

$$\begin{aligned} ^{\circ}(+2 + +3) &= ^{\circ}(+2) + ^{\circ}(+3) \\ &= -2 + -3 \\ &= -5 \end{aligned}$$

Would these two different procedures *always* yield the same result? How could you write this, using variables?

(5) The "rainbow picture" (so named by some children in Weston, Connecticut) looks like this:



The rule for using the "rainbow picture" is: "to find the additive inverse of a number, just go to the other end of the rainbow."

(6) Debbie is correct.

(7) (a) This is an identity.

You may want to view the film "Second Lesson," in which this occurs and is discussed.

- (b) Not an identity.  
 (c) Not an identity.

This is included, however, because sometimes the children like to "fix it up" so that it will be an identity. This occurs, for example, in the film "Second Lesson."

- (d)  $\square + \square = 2 \times \square$
- (e)  ${}^{\circ}(\square + \triangle) = {}^{\circ}\square + {}^{\circ}\triangle$
- (f)  ${}^{\circ}(\square \times \triangle) = ({}^{\circ}\square) \times ({}^{\circ}\triangle)$

- (d) This is an identity.
- (e) This is an identity.
- (f) This is not an identity. You can easily show that it is not. For example, make numerical replacements for the variables:

$$\begin{aligned} {}^+2 &\rightarrow \square \\ {}^+3 &\rightarrow \triangle \\ {}^{\circ}(\square \times \triangle) &= ({}^{\circ}\square) \times ({}^{\circ}\triangle) \\ {}^{\circ}({}^+6) &= ({}^+2) \times ({}^+3) \\ -6 &= {}^+6 \quad (\text{which is false}) \end{aligned}$$

(Recall that  ${}^+2 \times {}^+3 = {}^+6$ , as we have seen in our work with "post-man stories.")

- (g)  ${}^{\circ}({}^{\circ}\square) = \square$
- (h)  ${}^{\circ}[{}^{\circ}({}^{\circ}\square)] = \square$
- (i)  ${}^{\circ}[{}^{\circ}({}^{\circ}\square)] = {}^{\circ}\square$

- (g) This is an identity.
- (h) This is not an identity.
- (i) This is an identity.

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(8) Dan says you can change subtraction into addition by using the identity

$$\square - \triangle = \square + {}^{\circ}\triangle.$$

What do you think?

(9) We have seen how "additive inverses" and "subtraction" work. Do you know how "multiplicative inverses" work?

(8) Dan is correct.

(9) Now we begin to see some of the payoff from this "inverse" approach. Once we have attended to subtraction in class, the children can go on and work out most of the treatment of division by themselves!

First, however, we had best take a careful look at how we handled subtractions. To find the additive inverse of, say,  ${}^+3$ , we wrote

$${}^+3 + \underline{\quad} = 0.$$

That is to say, we considered the *open sentence*

$${}^+3 + \square = 0,$$

and we looked for the (unique) element in the truth set for this open sentence.

Now, if we want to use an analogous approach for division and multiplication, we must translate

$${}^+3 + \square = 0$$

into *multiplicative* terms. In the first place, zero played a special role in addition—as some of our children have said, "zero is the unchanger":

$$\begin{aligned} 3 + 0 &= 3 \\ 4 + 0 &= 4 \\ 5 + 0 &= 5 \\ &\vdots \\ &\vdots \end{aligned}$$

Is there an analogous "unchanger" in *multiplication*? In fact, there is. Consider

$$\begin{aligned} 3 \times 1 &= 3 \\ 4 \times 1 &= 4 \\ 5 \times 1 &= 5 \\ &\vdots \end{aligned}$$

Evidently, for *multiplication*, the "unchanger" is *one*. We can begin our translation:

$$\begin{aligned} {}^*3 + \square &= 0 \\ &\downarrow \\ &= 1 \end{aligned}$$

Presumably, we want to translate the + into  $\times$ :

$$\begin{aligned} {}^*3 + \square &= 0 \\ \downarrow &\quad \downarrow \\ \times &= 1 \end{aligned}$$

Hence, if we seek the *multiplicative inverse* of  ${}^*3$ , we should consider the open sentence  ${}^*3 \times \square = 1$ . The truth set, evidently, is  $\{\frac{1}{3}\}$ . Consequently, we say that the *multiplicative inverse* of  ${}^*3$  is  $\frac{1}{3}$ .

Another word, meaning the same thing as "multiplicative inverse," is *reciprocal*.

We can now define *division*. Let us use the symbol  ${}^r({}^*3)$  to denote the reciprocal of  ${}^*3$ , and  ${}^rA$  to denote the reciprocal of  $A$ , etc. Then we shall define  $A \div B$  to mean  $A \times {}^rB$ .

Notice the parallel:  $A - B$  means  $A + {}^cB$ ;  $A \div B$  means  $A \times {}^rB$ .

But wait! There is one small complication! If we try to find  ${}^r0$ , the reciprocal of zero, we must consider the open sentence  $0 \times \square = 1$ . It is always true that "zero times anything equals zero," that is to say, we have the *identity*  $\square \times 0 = 0$ .

Because of the commutative law for multiplication,  $0 \times \square = \square \times 0$ , and hence, if we must always have (as we must)  $\square \times 0 = 0$ , then we must also have  $0 \times \square = 0$ , and, since (in our usual arithmetic)  $0 \neq 1$ , we *cannot* have  $0 \times \square = 1$ .

Therefore, the truth set for the open sentence  $0 \times \square = 1$  is *the empty set*. It has no elements in it, and when we search around in it, seeking  ${}^r0$ , there is nothing available. In fact, *there is no multiplicative inverse of zero*.

This means that *we can never divide by zero*. Let us try to, and see what happens. If we want to find  $4 \div 0$ , we know that means, really,  $4 \times {}^r0$ . But  ${}^r0$  *does not exist*; consequently, the division  $4 \div 0$  cannot be performed. This difficulty will always arise whenever we try to divide by zero. In fact, *division by zero is always impossible*.

This discussion is far more precise than the usual discussions of "division by zero." It shows us far more clearly wherein the limitation lies—and, incidentally, it shows us how we would have to change our mathematical system if we wished to create a new mathematical system within which division by zero would be possible!

Everything that we have said here will extend, very nicely, to other, more abstract, algebraic systems which we shall encounter in the future (for example, it will extend to the algebra of "matrices," as we shall see later in this book).

(10) Debbie says that the "multiplicative inverse" of  $\square$  is "the number that I can multiply  $\square$  by to get 0." What do you think?

(11) Roger thinks Debbie is wrong. He says that the "multiplicative inverse" of  $\square$  is "the number you multiply  $\square$  by to get 1." Do you agree?

(12) Can you find the truth set for each of these open sentences?

(a)  $\cdot 2 \times \square = 0$

(b)  $\cdot 2 \times \square = 1$

(c)  $\cdot \frac{2}{3} \times \square = 1$

(d)  $\cdot \frac{2}{3} \times \square = 0$

(e)  $\cdot 2\frac{1}{2} \times \square = 1$

(f)  $0 \times \square = 1$

(13) Roger says that mathematicians call the "multiplicative inverse" of a number  $\square$  the "reciprocal" of  $\square$ , and that they write  $\ulcorner \square$ .

What is  $\ulcorner 7$ ? What is  $\ulcorner (\frac{1}{4} + \frac{1}{2})$ ?

(14) Which of these are identities?

(a)  $\square \times \ulcorner \square = 1$

(b)  $\square \times \ulcorner \square = 1$

(c)  $(\square + \triangle) \times \ulcorner (\square + \triangle) = 1$

(d)  $(\square + \triangle) \times (\ulcorner \square + \ulcorner \triangle) = 1$

(e)  $\ulcorner (\ulcorner \square) = \square$

(f)  $\ulcorner [\ulcorner (\ulcorner \square)] = \square$

(g)  $\ulcorner [\ulcorner (\ulcorner \square)] = \ulcorner \square$

(15) Can you change division into multiplication by using a "reciprocal"?

(16) Can you write a complete list of the axioms that we seem to be using thus far? Do you suppose this list is final? Will we ever want to change it?

(10) **Debbie is wrong. This question is meant to point the students' attention to "multiplicative inverses."**

(11) **Roger is correct. See the discussion in the answer to question 9.**

(12) (a)  $\{0\}$

(b)  $\{\frac{-1}{2}\}$

(c)  $\{\frac{13}{2}\}$

(d)  $\{0\}$

(e)  $\{\frac{14}{9}\}$

(f) **There is no number in the truth set of this open sentence since whatever number is inserted in the box will result in the sentence**

$$0 = 1$$

**(which is false). See the discussion in the answer to question 9.**

(13) **Roger is correct. See the discussion in the answer to question 9.**

$$\ulcorner 7 = \frac{1}{7} \quad \text{since} \quad 7 \times \frac{1}{7} = 1.$$

$$\ulcorner (\frac{1}{4} + \frac{1}{2}) = \ulcorner (\frac{3}{4}) \\ = \frac{4}{3}$$

(14) (a) Identity.

(b) Not an identity.

(c) Identity.

(d) Not an identity.

**You can easily show that it is not. For example, make numerical replacements for the variables:**

$$\cdot 2 \rightarrow \square$$

$$\cdot 4 \rightarrow \triangle$$

$$(\cdot 2 + \cdot 4) \times (\ulcorner \cdot 2 + \ulcorner \cdot 4) = 1$$

$$(\cdot 6) \times (\ulcorner \frac{1}{2} + \ulcorner \frac{1}{4}) = (\cdot 6) \times (\ulcorner \frac{3}{4}) = \frac{18}{4} = \frac{9}{2} = 1$$

**(which is false)**

(e) Identity.

(f) Not an identity.

(g) Identity.

(15) **Yes, see the discussion in the answer to question 9.**

(16) **An appropriate list of axioms is given at the beginning of Chapter 23, in the *Student Discussion Guide*. This list (presumably) is not final, but is the result of our study and understanding of algebra thus far. We might reasonably anticipate that as we study further, we shall want to modify this list of axioms.**

## PRACTICE IN MAKING UP YOUR OWN DERIVATIONS



### CHAPTER 23

## Practice in Making Up Your Own Derivations

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In this chapter, we want to get some practice in making up our own derivations.

To start with, we'll need a list of axioms. Let's agree to use this list, at least for the time being:

$$\square = \square$$

Reflexive Property of Equality (RPE)

$$\square + \triangle = \triangle + \square$$

Commutative Law for Addition (CLA)

$$\square \times \triangle = \triangle \times \square$$

Commutative Law for Multiplication  
(CLM)

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

Distributive Law (DL)

$$\square + (\triangle + \nabla) = (\square + \triangle) + \nabla$$

Associative Law for Addition (ALA)

$$\square \times (\triangle \times \nabla) = (\square \times \triangle) \times \nabla$$

Associative Law for Multiplication  
(ALM)

$$\square + 0 = \square$$

Addition Law for Zero (ALZ)

$$\square \times 0 = 0$$

Multiplication Law for Zero (MLZ)

$$\square \times 1 = \square$$

Law for 1 (L1)

$$\left. \begin{array}{l} 1 + 1 = 2 \\ 2 + 1 = 3 \\ 3 + 1 = 4 \\ \vdots \end{array} \right\}$$

Definition of the Numerals 2,  
3, 4, ... (Def. Num.)

## ANSWERS AND COMMENTS

$$\left. \begin{array}{l} +1 = 1 \\ +2 = 2 \\ +3 = 3 \\ \vdots \end{array} \right\} \text{Definition of the Numerals } +1, +2, \dots$$

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Every number has an additive inverse. If we use the notation  $^{\circ}(+2)$  to mean "the additive inverse of +2," then  $^{\circ}(+2)$  is defined as the number we must add to +2 in order to get zero.

In general,

$$\square + {}^{\circ}\square = 0$$

Law of Opposites (L. Opp.)

$$\square - \triangle = \square + {}^{\circ}\triangle$$

Definition of Subtraction (Def. Subtr.)

$$\left. \begin{array}{l} {}^{\circ}(+1) = -1 \\ {}^{\circ}(+2) = -2 \\ {}^{\circ}(+3) = -3 \\ \vdots \end{array} \right\} \text{Definition of the Numerals } -1, -2, -3, \dots$$

Every number except zero has a "reciprocal," or "multiplicative inverse." If we write

$${}^{\prime}2$$

to mean the "reciprocal of 2," then

$$\begin{aligned} {}^{\prime}2 &= \frac{1}{2} \\ 2 \times {}^{\prime}2 &= 1, \end{aligned}$$

and, in general,

$$\square \times {}^{\prime}\square = 1, \quad 0 \not\rightarrow \square^*$$

Law for Reciprocals (L. Recip.)

$$\square \div \triangle = \square \times {}^{\prime}\triangle, \quad 0 \not\rightarrow \triangle$$

Definition of Division (Def. Div.)

The list above gives us a reasonable set of axioms for our "algebra." For our "logic," we shall have two rules: PN and UV.

Now let's see if we can make up derivations.

Can you write a derivation for each theorem, using Marjory's method of writing?

$$(1) A \times (B + C) = A \times (C + B)$$

In the derivations given below, we use the "underlining" notation to indicate uses of PN. Some people find this helpful, especially when they are writing out a proof on a chalkboard, where one can watch the *order* in which things are written. If you do *not* find this helpful, I suggest you merely ignore the underlining.

(1) Here is one possible derivation:

$$(i) \underline{\square} = \square \quad \text{RPE}$$

$$(ii) A \times (B + C) = A \times \underline{(B + C)} \quad \text{UV: } A \times (B + C) \rightarrow \square$$

in line (i)

\*This notation is meant to remind us that, in the axiom  $\square \times {}^{\prime}\square = 1$ , zero must never be used as a replacement for the variable  $\square$ .

(2)  $A \times (B + C) = (B + C) \times A$

(3)  $A \times (B + C) = (C \times A) + (B \times A)$

(iii)  $\square + \triangle = \triangle + \square$

(iv)  $B + C = C + B$

(v)  $A \times (B + C) = A \times \underline{(C + B)}$

Q.E.D.

(2) (i)  $\square \times \triangle = \triangle \times \square$

(ii)  $A \times (B + C) = (B + C) \times A$

Q.E.D.

(3) (i)  $\square \times (\triangle + \nabla)$   
 $= (\square \times \triangle) + (\square \times \nabla)$

(ii)  $A \times (B + C)$   
 $= \underline{(A \times B)} + (A \times C)$

(iii)  $\square + \triangle = \triangle + \square$

(iv)  $(A \times B) + (A \times C)$   
 $= (A \times C) + (A \times B)$

(v)  $A \times (B + C)$   
 $= \underline{(A \times C)} + (A \times B)$

(vi)  $A \times (B + C)$   
 $= \underline{(A \times C)} + (A \times B)$

(vii)  $\square \times \triangle = \triangle \times \square$

(viii)  $A \times C = C \times A$

(ix)  $A \times (B + C)$   
 $= \underline{(C \times A)} + (A \times B)$

(x)  $A \times (B + C)$   
 $= (C \times A) + \underline{(A \times B)}$

CLA

UV:  $B \rightarrow \square$

$C \rightarrow \triangle$   
 in line (iii)

PN, from line (ii),  
 using line (iv)

CLM

UV:  $A \rightarrow \square$

$B + C \rightarrow \triangle$   
 in line (i)

DL

UV:  $A \rightarrow \square$

$B \rightarrow \triangle$

$C \rightarrow \nabla$

in line (i)

CLA

UV:  $A \times B \rightarrow \square$

$A \times C \rightarrow \triangle$

in line (iii)

PN, from line (ii),  
 using line (iv)

Repeat of line (v),  
 in order to avoid  
 confusing the under-  
 lining

CLM

UV:  $A \rightarrow \square$

$C \rightarrow \triangle$

in line (vii)

PN, from line (vi),  
 using line (viii)

Repeat of line (ix), in  
 order to avoid  
 confusing the under-  
 lining for two dif-  
 ferent applica-  
 tions  
 of PN



(xi)  $A \times B = B \times A$

UV:  $A \rightarrow \square$

$B \rightarrow \triangle$   
in line (vii)

(xii)  $A \times (B + C)$   
 $= (C \times A) + \underline{(B \times A)}$

PN from line (x),  
using line (xi)

Q.E.D.

(4) Lex made up a derivation for the theorem

$$A + (B \times C) = (C \times B) + A.$$

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Cynthia complained that she couldn't understand Lex's derivation, so Bob tried to describe it.

Lex's derivation                      Bob's description

$$A + (B \times C) = A + (B \times C)$$

Actually, Lex really began with RPE:

$$\square = \square.$$

Lex knew this was an axiom. Then  
Lex used UV

UV:  $A + (B \times C) \rightarrow \square$

to get

$$A + (B \times C) = A + (B \times C).$$

$$A + (B \times C) = A + (C \times B)$$

Now, Lex used PN. He erased the name

$$B \times C,$$

to get

$$A + (B \times C) = A + (\overset{\uparrow}{\text{gaping hole}}).$$

Then, into this gaping hole, he put  
another name for the same thing.

$$A + (B \times C) = A + (C \times B).$$

How did Lex know that  $C \times B$  named  
the same thing that  $B \times C$  did? He used  
CLM

$$\square \times \triangle = \triangle \times \square$$

and used UV

UV:  $B \rightarrow \square$   
 $C \rightarrow \triangle$

to get

$$B \times C = C \times B,$$

which says that  $C \times B$  names the same  
thing that  $B \times C$  does.

$$A + (B \times C) = (C \times B) + A$$

Here, Lex again used PN. He began  
with the identity

$$A + (B \times C) = A + (C \times B).$$

(4) This is really not a "question" to be answered, but rather a section to be read by each student, and then discussed (as much as necessary) by the class.

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He erased the entire right-hand side,

$$A + (B \times C) = \text{██████████}$$

↑  
(Gaping hole)

and into the gaping hole he put another name for the same thing,

$$(C \times B) + A,$$

to get

$$A + (B \times C) = (C \times B) + A.$$

How did Lex know that

$$(C \times B) + A$$

named the same thing that

$$A + (C \times B)$$

named? The answer is that he started with CLA

$$\square + \triangle = \triangle + \square,$$

and used UV

$$UV: (C \times B) \rightarrow \square$$

$$A \rightarrow \triangle$$

to get

$$(C \times B) + A = A + (C \times B),$$

which says that

$$(C \times B) + A$$

names the same thing that

$$A + (C \times B)$$

names.

Do you understand Bob's description of what Lex did?

Can you write out a derivation for each theorem, using Debbie's method?

(5) Theorem:  $\square + \square = 2 \times \square$

In the derivation below, we introduce the symbol  $A$  to indicate one variable, in order to avoid the confusion that might result if we had too many *different* uses of the symbol  $\square$ .

(5) There are many possible approaches. Two excellent original ones, by a fifth-grade class, can be studied from the tape recording #D-1, available from the Madison Project.

(i)  $\square = \square$

(ii)  $2 \times A = 2 \times A$

(iii)  $\square \times \triangle = \triangle \times \square$

(iv)  $2 \times A = A \times 2$

RPE

UV:  $2 \times A \rightarrow \square$   
in line (i)

CLM

UV:  $2 \rightarrow \square$

$A \rightarrow \triangle$   
in line (iii)

- (v)  $\underline{A \times 2} = 2 \times A$  PN, from line (ii), using line (iv)
- (vi)  $A \times \underline{2} = 2 \times A$  Repeat of line (v)
- (vii)  $2 = 1 + 1$  Def. Num.
- (viii)  $A \times \underline{(1 + 1)} = 2 \times A$  PN, from line (vi), using line (vii)
- (ix)  $\square \times (\triangle + \nabla)$  DL  
 $= (\square \times \triangle) + (\square \times \nabla)$
- (x)  $A \times (1 + 1)$  UV:  $A \rightarrow \square$   
 $= (A \times 1) + (A \times 1)$   $1 \rightarrow \triangle$   
 $1 \rightarrow \nabla$   
in line (ix)
- (xi)  $\underline{A \times (1 + 1)} = 2 \times A$  Repeat of line (viii)
- (xii)  $\underline{(A \times 1) + (A \times 1)} = 2 \times A$  PN, from line (xi), using line (x)
- (xiii)  $\square \times 1 = \square$  L1
- (xiv)  $A \times 1 = A$  UV:  $A \rightarrow \square$   
in line (xiii)
- (xv)  $\underline{(A \times 1) + (A \times 1)} = 2 \times A$  Repeat of line (xii)
- (xvi)  $\underline{A} + (A \times 1) = 2 \times A$  PN, from line (xv), using line (xiv)
- (xvii)  $A + \underline{(A \times 1)} = 2 \times A$  Repeat of line (xvi)
- (xviii)  $A + \underline{A} = 2 \times A$  PN, from line (xvii), using line (xiv)
- (xix)  $\square + \square = 2 \times \square$  UV:  $\square \rightarrow A$   
in line (xviii)

Q.E.D.

(6) Theorem:  $A + (B + C) = C + (B + A)$

- (i)  $\square + (\triangle + \nabla)$  ALA  
 $= (\square + \triangle) + \nabla$
- (ii)  $A + (B + C) = \underline{(A + B)} + C$  UV:  $A \rightarrow \square$   
 $B \rightarrow \triangle$   
 $C \rightarrow \nabla$   
in line (i)
- (iii)  $\square + \triangle = \triangle + \square$  CLA
- (iv)  $A + B = B + A$  UV:  $A \rightarrow \square$   
 $B \rightarrow \triangle$   
in line (iii)

(v)  $A + (B + C) = \underline{(B + A)} + C$  PN, from line (ii),  
using line (iv)

(vi)  $A + (B + C) = \underline{(B + A)} + C$  Repeat of line (v)

(vii)  $(B + A) + C = C + (B + A)$  UV:  $B + A \rightarrow \square$   
 $C \rightarrow \triangle$   
in line (iii)

(viii)  $A + (B + C) = \underline{C + (B + A)}$  PN, from line (vi),  
using line (vii)

**Q.E.D.**

(7) Theorem:  $3 + 2 = 5$

(7) (i)  $\square = \square$

RPE

(ii)  $3 + 2 = 3 + \underline{2}$

UV:  $3 + 2 \rightarrow \square$   
in line (i)

(iii)  $2 = 1 + 1$

Def. Num.

(iv)  $3 + 2 = 3 + \underline{(1 + 1)}$

PN, from line (ii),  
using line (iii)

(v)  $\square + (\triangle + \nabla)$   
 $= (\square + \triangle) + \nabla$

ALA

(vi)  $3 + (1 + 1) = \underline{(3 + 1)} + 1$

UV:  $3 \rightarrow \square$   
 $1 \rightarrow \triangle$   
 $1 \rightarrow \nabla$   
in line (v)

(vii)  $3 + 2 = \underline{3 + (1 + 1)}$

Repeat of line (iv)

(viii)  $3 + 2 = \underline{(3 + 1)} + 1$

PN, from line (vii),  
using line (vi)

(ix)  $3 + 1 = 4$

Def. Num.

(x)  $3 + 2 = \underline{(3 + 1)} + 1$

Repeat of line (viii)

(xi)  $3 + 2 = \underline{4} + 1$

PN, from line (x),  
using line (ix)

(xii)  $4 + 1 = 5$

Def. Num.

(xiii)  $3 + 2 = \underline{4 + 1}$

Repeat of line (xi)

(xiv)  $3 + 2 = \underline{5}$

PN, from line (xiii),  
using line (xii)

**Q.E.D.**

(8) Theorem:  $(A + B) + (C + D)$   
 $= [D + (C + B)] + A$

(8) (i)  $\square = \square$

RPE

(ii)  $(A + B) + (C + D)$   
 $= \underline{(A + B)} + (C + D)$

UV:  $(A + B)$   
 $+ (C + D) \rightarrow \square$   
in line (i)

- (iii)  $\square + (\triangle + \nabla)$   
 $= (\square + \triangle) + \nabla$  ALA
- \*(iv)  $A + [B + (C + D)]$  UV:  $A \rightarrow \square$   
 $= (A + B) + (C + D)$   $B \rightarrow \triangle$   
 $C + D \rightarrow \nabla$   
in line (iii)
- (v)  $(A + B) + (C + D)$  PN, from line (ii),  
 $= A + [B + (C + D)]$  using line (iv)
- ! (vi)  $B + (C + D) = (B + C) + D$  UV:  $B \rightarrow \square$   
 $C \rightarrow \triangle$   
 $D \rightarrow \nabla$   
in line (iii)
- (vii)  $(A + B) + (C + D)$  Repeat of line (v)  
 $= A + [B + (C + D)]$
- (viii)  $(A + B) + (C + D)$  PN, from line (vii),  
 $= A + [(B + C) + D]$  using line (vi)
- (ix)  $(A + B) + (C + D)$  Repeat of line (viii)  
 $= A + [(B + C) + D]$
- (x)  $\square + \triangle = \triangle + \square$  CLA
- (xi)  $B + C = C + B$  UV:  $B \rightarrow \square$   
 $C \rightarrow \triangle$   
in line (x)

We shall finish this using a shorter notation:

- (xii)  $(A + B) + (C + D)$  PN, from line (ix),  
 $= A + [(C + B) + D]$  using line (xi)
- (xiii)  $(A + B) + (C + D)$  CLA  
 $= A + [D + (C + B)]$
- (xiv)  $(A + B) + (C + D)$  CLA  
 $= [D + (C + B)] + A$

Q.E.D.

\*This use of ALA is very common: one often has four terms (here:  $A, B, C, D$ ) to fit into the three frames  $\square, \triangle, \nabla$ . Since it is  $A$  which we wish to isolate by itself (see the right-hand side of the identity we are trying to obtain), we use UV as we do here.

! In this step, we wish to isolate  $D$ . This fact gives us our best hint as to how to use UV in ALA.

(9) Theorem:  $(A + B) \times (A + B)$   
 $= \{(A \times A) + [(B + B) \times A]\}$   
 $+ (B \times B)$

(9) For this problem, we shall use the shorter method of writing:

$$\begin{aligned} (A + B) \times (A + B) &= (A + B) \times (A + B) && \text{RPE} \\ & && \text{DL} \\ (A + B) \times (A + B) &= [(A + B) \times A] && \swarrow \\ &+ [(A + B) \times B] && \text{CLM}^{2*} \\ (A + B) \times (A + B) &= [A \times (A + B)] && \swarrow \\ &+ [B \times (A + B)] && \text{DL}^2 \\ (A + B) \times (A + B) &= [(A \times A) + (A \times B)] && \swarrow \\ &+ [(B \times A) + (B \times B)] && \text{ALA} \\ (A + B) \times (A + B) &= (A \times A) + \{(A \times B) && \swarrow \\ &+ [(B \times A) + (B \times B)]\} && \text{ALA} \\ (A + B) \times (A + B) &= (A \times A) + \{[(A \times B) && \swarrow \\ &+ (B \times A)] + (B \times B)\} && \text{CLM} \\ (A + B) \times (A + B) &= (A \times A) + \{[(A \times B) && \swarrow \\ &+ (A \times B)] + (B \times B)\} && \text{DL} \\ (A + B) \times (A + B) &= (A \times A) && \swarrow \\ &+ \{[A \times (B + B)] + (B \times B)\} && \text{ALA} \\ (A + B) \times (A + B) &= \{(A \times A) && \swarrow \\ &+ [A \times (B + B)]\} + (B \times B) && \text{CLM} \\ (A + B) \times (A + B) &= \{(A \times A) && \swarrow \\ &+ [(B + B) \times A]\} + (B \times B) && \end{aligned}$$

Q.E.D.

\*The exponent 2 means that the axiom has been used twice in the step taken.

## EXTENDING SYSTEMS: “LATTICES ” AND EXPONENTS

We put this chapter in at this point in order to provide some background in exponents, which will be helpful in following chapters. While we are at it, however, we may as well kill two birds with one chapter, so we are including another matter of considerable importance: once people recognize the creative way in which men build mathematical systems, they soon realize also that, like medieval cathedrals and New York City, mathematical systems are often unfinished and still in a state of being elaborated and extended. One comes now and then to the “frontier” or “growing edge” of the system, and needs to build further in order to move ahead.

Now, when this happens, there is a delightful interplay of creative freedom and originality on the one hand, and the restrictive logic of the previously existing structure on the other hand. You can see this interplay at work in art, music, architecture, and literature, as well.

In general, when you are writing the third act of a play, you have an important role for creative originality and freedom—a very large role, indeed! Yet you are bound, in a subtle way, by the logic of the first two acts. You should, in general, have the same characters in the third act as in the first two. The unexplained appearance or disappearance of characters between acts two and three would ordinarily be considered a weakness (unless, as in Kafka, the logic of the preceding acts has already accustomed the audience to such mysterious appearances and disappearances). Moreover, each individual character should behave in a way that is consistent with the personality you have established for him in the previous acts. The important thematic elements of the first two acts should, ordinarily, be carried on into the third act. You can find for yourself many other respects in which the third act is expected to “grow naturally” out of the first two.

So much for plays. Let us turn now to mathematics, and see how these encounters at the “frontiers” lead us to extend our mathematical system in a way that is consistent with the already existing part of the structure.

The two mathematical structures we have chosen to study are both interesting in their own right. One, the system of exponents, was gradually elaborated over a period of centuries. Descartes (1596-1650) made use of exponents on the level of elaboration of

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243,$$

and also with *variables*, such as

$$x^3 = x \cdot x \cdot x$$

or what we should write as

$$\square^3 = \square \times \square \times \square$$

(although, of course, Descartes did not himself use the symbol  $\square$  to denote a variable). Earlier versions of exponents can be

found, in various forms, for example in the writing of Nicole Oresme (1312-1382). See Appendix A: Newman (158), Eves (151).

An extension of the system of exponents, to include zero, negative, and fractional exponents, very much in the spirit of the present chapter of *Explorations*, was given by the great English mathematician John Wallis (1616-1703). Thus, it would be a modest oversimplification, but perhaps reasonably accurate nonetheless, to say that the "simple" part of the structure of exponents—using positive integers—was the work of the first half of the seventeenth century (indeed, partly the work of our old friend Descartes), while the systematic *extension* of this system "beyond the frontiers"—i.e., into zero, negative, and fractional exponents—was the work of the later half of the seventeenth century (in the writing of the Englishman John Wallis).

The other mathematical system which we shall consider is far more recent. It is the creation of Professor David Page of the University of Illinois, and is one of the original mathematical explorations which Professor Page worked out in the classroom, as a joint effort between himself and his class of elementary school children. It dates from the 1950's and 1960's.

We shall now develop these two systems. In each case, we shall develop the "basic" or "simple" system first; then we shall run into the "frontier," where the structure is not complete; and then we shall extend the structure, so as to be consistent with what had previously been built.



## CHAPTER 24

## Extending Systems:

## "Lattices" and Exponents

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Frequently, we build up a mathematical system for some reason or other, and are proud of it because it is our own creation and because it seems to "work."

Then, on some black day or other, we discover that our system does not "work" any longer. We reach a point that our system cannot cope with. (This is somewhat like the feeling people had, before Columbus, that if they came to the edge of the world they would fall off. We have come to the "edge" of our beautiful mathematical system, and we seem to be in danger of having nowhere else to go.)

Can we build on to our system? Can we extend it further?

Let's look at this, in two important cases.

## I. THE SYSTEM OF "LATTICES"

Professor David Page, of the University of Illinois, has introduced an interesting mathematical system, which we might represent in the following way.

## ANSWERS AND COMMENTS



To begin with, we write numbers in an array or "lattice" like this:

↑ (and so on)

31	32								
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

Now, this gives us a new way to write names for numbers:

(1) What number do you suppose is meant when we write

$$3 \rightarrow ?$$

(2) What number do you suppose is meant when we write

$$7 \uparrow ?$$

Can you find simpler names for each of these numbers?

(3)  $8 \rightarrow$

(4)  $9 \leftarrow$

(5)  $5 \uparrow \uparrow \uparrow$

(6)  $3 \nearrow$

(7)  $9 \leftarrow \uparrow \leftarrow$

(8)  $21 \downarrow \rightarrow$

(9)  $3 \uparrow \uparrow \uparrow \uparrow \uparrow$

(10)  $3 \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$

(11)  $24 \swarrow$

(12)  $26 \swarrow \rightarrow$

(13)  $27 \swarrow \rightarrow \uparrow$

(14)  $27 \swarrow \rightarrow \uparrow \uparrow$

(15)  $27 \swarrow \rightarrow \uparrow \rightarrow$

(16)  $27 \rightarrow \rightarrow \uparrow \swarrow$

(17)  $27 \uparrow \swarrow \rightarrow \rightarrow$

(18)  $27 \uparrow \rightarrow \swarrow \rightarrow$

First, we build up the "simple" or "basic" part of the structure. At this stage, Professor Page deliberately and wisely operates *on an intuitive level only*. He refuses to explain "how" he is doing these problems! To offer any explanation *at this stage* would make it nearly impossible for the children to use any creative originality in extending the system later on. Each child is invited to guess how *he* thinks these problems should be handled. Incidentally, it is probably a good idea to write the array

41	42	43							
31	32	33	34	35	36	37	38	39	40
21	22	23	24	25	26	27	28	29	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

on the chalkboard at the front of the room, and keep it there during this entire discussion. But—*at this stage*—do *not* show how you are using this array!

(1)  $3 \rightarrow$  names the same number that 4 does:

$$3 \rightarrow = 4.$$

(2)  $7 \uparrow$  names the same number that 17 names. Recalling the meaning of the symbol  $=$ , we can write

$$7 \uparrow = 17.$$

(3) 9

(4) 8

(5) 35

(6) 14

(7) 17

(8) 12

(9) 53

(10) 13

(11) 13

(12) 16

(13) 27

(14) 37

(15) 28

(16) 28

(17) 28

(18) 28

Notice that in these problems we are *not* telling the students how to interpret the arrows—we are merely telling them the results of using the arrows. In particular, we are *not* telling them to interpret the arrows as "motions" on the array of numbers! (Actually, it is virtually certain that the students *are* interpreting the arrows in this way, but we are not allowing anyone to say so explicitly, because we shall soon want to ask the children just what the arrows really do mean.)

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- (19) 27 ↑ → ↙ ← ←
- (20) 23 ↔ ↔ ↔ ↔ ↔ ↔
- (21) 14 ↔ ↔ ↔ ↔ ↔ ↔ ↑
- (22) 14 ↔ ↔ ↔ ↔ ↔ ↔ ←
- (23) 14 ↔ ↔ ↔ ↔ ↔ ↔ ↓
- (24) 12 ↗ ↗ ↗ ↓ ↓ ↓ ← ←

- (19) 25
- (20) 23
- (21) 24
- (22) 13
- (23) 4
- (24) 13

In questions 1 through 24, we have been developing, on an intuitive level, the "simple" or "basic" part of Professor Page's mathematical structure, which he refers to as a "lattice."

Now, with questions 25 through 30, we begin to arrive at the "frontiers" or "incompleted growing edges" of this structure. We have come to the edge of the world and are in danger of falling off.

In problems 25 through 30, we suggest you do *not* point out this danger to the students. They may see it, or they may not. Don't worry. The discomforts of frontier life will become apparent to everyone as soon as we reach problems 31 through 34, and that will be quite soon enough.

Of course, if a student *does* discover the frontier at this point, take full advantage of his contribution.

Note: Before working on question 25, it may be advisable to ask the class what an *identity* is. If your students have not noticed that they have now reached the edge of the world, they will (presumably) say that this *is* an identity. But if they *have* noticed the limitations of the mathematical structure we have been using, they may be much less certain.

Let us look at the difficulty. Since we have never, thus far, discussed with the children *how* they are handling these problems, there is every reason to hope that different children have been handling them in different ways.

Here are some ways they *may* have been using:

- (i) Since we have the geometric array of numbers in full view on the front board, and since the symbolism of arrows

3 ↑  
5 ↗  
25 ↙  
39 ↓ ←

(and so on) makes a deliberate appeal to one's intuitive notions of *motion*, we *hope* that most students are imagining actual moves around the array of numbers. For example, for 17 ↑ such students would be thinking of "moving" from 17, straight upward "one step," and consequently landing at 27.

. . .  
31 32 . . .  
21 22 23 24 25 26 27 28 29 30  
11 12 13 14 15 16 17 18 19 20  
1 2 3 4 5 6 7 8 9 10

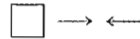
(ii) Some students may have done this. Having started out thinking in terms of geometrical motions, they may have gone on to notice that an "upward" arrow,  $\uparrow$ , seems to have the effect of "adding ten"; a "downward" arrow,  $\downarrow$ , seems (usually!) to have the effect of "subtracting ten"; an arrow pointing to the right,  $\rightarrow$ , seems (usually!) to have the effect of "adding one"; an arrow pointing to the left seems to have the effect of "subtracting one"; an arrow like  $\nearrow$  seems to be equivalent to two other arrows, namely  $\rightarrow \uparrow$ ; and so on.

If a student has recognized this *seemingly equivalent* reinterpretation of the arrow symbols, he may be making use of it. Indeed—partly unconsciously, perhaps—he may have discarded the "geometric movements" idea, and be handling these problems entirely arithmetically.

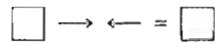
(iii) Still other student approaches are possible, but we leave you to discover them from your own children. The two above are perhaps the most basic.

Now, let's see what happens to these two kinds of student methods, when they encounter problems 25 through 35.

(i) Those students using the *geometric motions* approach may (if they don't notice the imminence of falling off the edge of the earth) reason that



means start somewhere on the lattice, move one step to the right, then move one step to the left. You get back where you started, so



is an identity.

You can try this out for yourself, using actual motions on the array, starting at 13, or 22, or 5, etc. Here is one. Start at 14:

⋮
⋮
⋮
31 32 33 ...
21 22 23 24 25 26 27 28 29 30
11 12 13 14 15 16 17 18 19 20
1 2 3 4 5 6 7 8 9 10

Move one step to the right:

⋮
⋮
⋮
31 ...
21 22 23 24 25 26 27 28 29 30
11 12 13 14 → 15 16 17 18 19 20
1 2 3 4 5 6 7 8 9 10

Then (since you have just arrived at 15), turn around and move one step to the left:

⋮
⋮
⋮
21 22 23 24 25 26 27 28 ...
11 12 13 14 ← 15 16 17 18 19 20
1 2 3 4 5 6 7 8 9 10

Behold! You *do* get back to 14. Consequently, the statement  $14 \rightarrow \leftarrow = 14$  is *true*.

*But—here is the crucial question! Will this always work?*

Suppose you start at 20. When you attempt to move one step to the right:

	...
31	...
21	22 23 24 25 26 27 28 29 30
11	12 13 14 15 16 17 18 19 20 →
1	2 3 4 5 6 7 8 9 10

you have nowhere to go! There *is* no number at  $20 \rightarrow$ . But, if  $20 \rightarrow$  is meaningless, then it may be risky to build onto the meaningless symbol  $20 \rightarrow$  in order to get  $20 \rightarrow \leftarrow$ . Hence, it would appear that  $20 \rightarrow \leftarrow$  is *meaningless*, and so the statement  $20 \rightarrow \leftarrow = 20$  can hardly be classified as either *true* or *false*!

This appears to be chaos. We have reached the edge of the world, and fallen off. Or, less metaphorically, we have gone as far as our mathematical system will let us go.

The question, now, is *can we extend our mathematical system?*

Before we attempt to do this, let us look first at how this problem appears to those students who have discarded geometrical motions, and are handling these problems arithmetically.

(ii) For the student who interprets  $\rightarrow$  as “add one,” and who interprets  $\leftarrow$  as “subtract one,” there seems to be no difficulty. The expression  $\square \rightarrow \leftarrow$  means merely “start with some number, then add one, then subtract one.”

This can never lead to difficulty, and will always get you back where you started. Consequently, *with this interpretation*,  $\square \rightarrow \leftarrow = \square$  does appear to be an identity.

Does this mean that, for these students, the “frontier” poses no problems? Unfortunately, no—not, at least, if they think carefully about the matter. For they *arrived* at their interpretation of  $\rightarrow$  as “add one” because they believed that  $\rightarrow$  *always* meant the same thing as “add one.”

But now they have just seen that this is not so! The name “20 plus 1” clearly refers to 21, but “ $20 \rightarrow$ ” does not name any number whatsoever, in the original geometrical sense. Hence, when these students replaced geometric motions with arithmetic operations, *they may not have been justified in doing so!*

Since the matter of *extending mathematical structures* is of very great importance in the study of mathematics, it is worth devoting some thought to this right now.

Many times a structure is built, and must sooner or later be extended.

Here are some examples:

(i) The number system 1, 2, 3, 4, ... must sooner or later be extended to include zero; i.e., it must be extended to the larger system 0, 1, 2, 3, 4, 5, ...

(ii) While there is choice as to *which* extension we shall make next, the system 0, 1, 2, 3, 4, 5, ... must sooner or later be extended still further. We might, for example, extend to the larger system ... -3, -2, -1, 0, +1, +2, +3, ...

(iii) The system  $\dots, ^{-3}, ^{-2}, ^{-1}, 0, ^{+1}, ^{+2}, ^{+3}, \dots$  must be extended sooner or later (if this was not already done earlier) to include *fractions*. (In this way, we arrive at the system of *rational numbers*: this includes all integers, positive and negative; zero; and all fractions, "mixed" numbers, terminating decimals, and repeating decimals, whether positive or negative.)

(iv) Two further extensions of this number system must be made sooner or later to get the *system of real numbers* and the *system of complex numbers*, but we shall not discuss these at this point.

(v) The system of *integer exponents*

$$\begin{aligned} 3^2 &= 3 \times 3 = 9 \\ 3^3 &= 3 \times 3 \times 3 = 27 \\ 3^4 &= 3 \times 3 \times 3 \times 3 = 81 \\ &\vdots \\ &\vdots \end{aligned}$$

must be extended to include *zero exponents*,  $3^0 = ?$ ; *negative exponents*,  $3^{-2} = ?$ ; and *fractional exponents*,  $3^{1/2} = ?$ . (Even further extensions, to *real* and *complex* exponents, must be made at a later stage in one's studies.)

(vi) The system of *factorials* for positive integers

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120 \\ &\vdots \\ &\vdots \end{aligned}$$

must be extended to allow for *zero factorial*,  $0! = ?$ ; for the *factorials of negative integers*,  $^{-1}! = ?$ ,  $^{-2}! = ?$ ; and for the *factorials of rational numbers*,  $\frac{1}{2}! = ?$ ,  $7\frac{3}{4}! = ? \dots$

This latter problem was finally—and beautifully—solved in the eighteenth century by Leonard Euler, one of the greatest mathematicians who has ever lived.

(vii) Many other examples could be given, such as Abelian summation, Cesaro summation, analytic continuation, and so on, but they are far beyond the scope of this book. Suffice it to say that *extending mathematical structures* is one of the *very* important problems that recurs throughout the study of mathematics.

In the present book, of course, we mean to look at this problem in two very simple cases only: Page's "lattices" and the system of exponents.

Having argued that extending systems is important, let's see how it works.

Speaking somewhat roughly and intuitively, we have a system that we have already built. For one reason or another, we bump into the edge of our system somewhere, and realize that it does, after all, have its undeveloped frontiers. What we usually do, then, is to take a very careful look at the way our already existing structure behaves. We then select among these attributes of the "original" system, and ask ourselves, *which of them are so important that we want to preserve them, that we want them to be attributes of our extended system as well?*

On the other hand, which are expendable, and can be allowed to fall by the wayside?

Once we have decided which attributes to preserve, we use our imagination to try to extend our structure in every way we can devise (at least until our time and patience run out).

Then we look at each extension in turn, and ask ourselves, if we extend the system *this* way, will the essential attributes still apply? If not, we discard that method of extending our system. If they *will* still apply, then we accept that method of extending our system – and, behold, we now have a “larger” system!

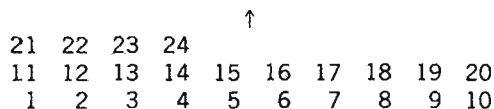
Let’s try it, on Pages’s lattice. When we get to  $20 \rightarrow$ , we bump into the undeveloped frontier of our system. We must find ways to extend the system. Here are some actual suggestions, mostly made by children (one or two, I must admit, were suggested by teachers):

(i) (Often suggested by children in grades 4 through 6.) Wrap the paper into a cylinder, so that after 20, you come around again to 11. With this extension, we will have  $20 \rightarrow = 11$ .

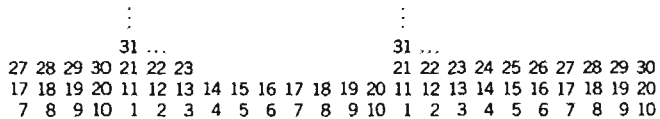
(ii) (Often suggested by children in grades 4 through 8.) Wrap the paper around into a kind of “barber-pole” cylinder, so that after 20 you come around and start the next higher line.

With this extension, we will have  $20 \rightarrow = 21$ .

(iii) (Often suggested by adults; sometimes by children.) Imagine that the array



is on a rubber stamp, and just keep stamping the rubber stamp down, to give a kind of wallpaper design:



With this extension, we will have  $20 \rightarrow = 11$ .

(iv) Since  $\rightarrow$  *sometimes* means “plus one,” pretend that it *always* does. In other words, throw away the geometry and fall back entirely on arithmetic. With this extension, we will have  $20 \rightarrow = 21$ .

(v) (Suggested by a teacher from New Hampshire, where it snows!) When you get to the edge and can’t go any further, just sit there and spin your wheels. With this extension, we get  $20 \rightarrow = 20$ .

(vi) (Suggested by a teacher from Manhattan, which is really a rather small island with rather a lot built up on it.) When you get to the edge and can’t move to the right, move up instead. With this extension, we get  $20 \rightarrow = 30$ .

Probably you and your students can think of yet other ways to extend our original system.

Now we have to choose among them! How shall we do it? Well, let’s see how well they work. How satisfactorily do these various



Adopt this extension only if you have a taste for asceticism, self-denial, and the long-hard-road approach. (But the choice is yours!)

System (vi). This system has the same general kind of complications that system (v) does.

Now that we know where we stand, let's return to the questions in the *Student Discussion Guide*.

Which of these open sentences are identities?

(25)  $\square \rightarrow \leftarrow = \square$

(26)  $\square \nearrow \downarrow \rightarrow = \square$

(27)  $\square \nearrow \downarrow \leftarrow = \square$

(28)  $\square \nearrow \nearrow \downarrow \downarrow \downarrow \leftarrow \leftarrow = \square$

(29)  $\square \nearrow \nearrow \downarrow \downarrow \downarrow \leftarrow \leftarrow = \square$

(30)  $\square \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \rightarrow \rightarrow \swarrow \swarrow \uparrow \uparrow = \square$

Can you find simpler names for each of these numbers?

(31)  $15 \uparrow$

(32)  $9 \rightarrow \rightarrow$

(25) (We can now discuss this more fully, at least among ourselves. The class discussion with the students should be allowed to grow naturally!)

If you don't notice the difficulties at the edges, then you will say that this is an identity. If you do notice the difficulties at the edges, then you realize that the question is meaningless until you make a suitable extension of your mathematical system! Whether the open sentence  $\square \rightarrow \leftarrow = \square$  turns out to be an identity will depend upon which extension you select. Quite a few different ones are possible.

(26) This question is similar to question 25. The motion  $\square \nearrow$  may cause you to "fall off the edge." Of course, in any "reasonable" extension,

$$\square \nearrow \downarrow \rightarrow = \square$$

will not turn out to be an identity.

(27) This question is similar to question 25. If, for example, you choose the "arithmetic" extension, then the open sentence

$$\square \nearrow \downarrow \leftarrow = \square$$

will be an identity. (Notice also the "closed triangle" pattern of  $\nearrow \downarrow \leftarrow$ .)

(28) Not an identity.

(29) Similar to question 25.

(30) Similar to question 25.

Some students (and some classes) may not have detected the pitfalls hidden in questions 25 through 30. For these students we now make the issue sharper and clearer. Questions 32 through 34 are intended to cause every student to notice the "falling off the edge" phenomenon.

(31) 25 (No difficulty here!)

(32) Here we encounter the frontier. The symbol  $9 \rightarrow \rightarrow$  means, of course,  $(9 \rightarrow) \rightarrow$ , which we can analyze by using a form of PN. The symbol  $(9 \rightarrow) \rightarrow$  names the same thing as  $10 \rightarrow$  does. But, unfortunately, if we have not extended our original system, the symbol  $10 \rightarrow$  doesn't mean anything at all!

If your class has not already extended the original system, this might be a good time to do so. By now the need should be apparent!



(33)  $12 \checkmark \leftarrow$

(34)  $22 \leftarrow \leftarrow$

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(35) Doug's teacher wrote this on the chalkboard:

In our system I tried to give Doug  
 some more study in the structure  
 of Professor Page's system. The  
 definition of a mathematical  
 system is so important, and I wish  
 that I could have had more active  
 criticism.

We have just had an example of  
 the danger of Professor Page's  
 system, and we need to extend  
 it. We found several different  
 systems, and I will write down  
 some that I liked best.

Did you find a "natural" extension that did not destroy the structure of Professor Page's system?

(36) Did you finally settle on any definite way, that everyone in class agreed with, to extend Professor Page's lattice?

## II. THE SYSTEM OF EXPONENTS

We frequently encounter problems like the following.

$$\begin{aligned} 2 \times 2 &= 4 \\ 2 \times 2 \times 2 &= 8 \\ 2 \times 2 \times 2 \times 2 &= 16 \\ 3 \times 3 &= 9 \\ 3 \times 3 \times 3 &= 27 \\ 3 \times 3 \times 3 \times 3 &= 81 \\ 10 \times 10 &= 100 \\ 10 \times 10 \times 10 &= 1000 \end{aligned}$$

(33)  $12 \checkmark \leftarrow = (12 \checkmark) \leftarrow = 1 \leftarrow$

That is to say (recalling what  $=$  means!), the symbol  $12 \checkmark \leftarrow$  names the same thing that  $1 \leftarrow$  names—but, unfortunately,  $1 \leftarrow$  doesn't name anything at all. It is a meaningless symbol—*unless we have extended our original system.*

(If we use the "arithmetic" extension, of course  $1 \leftarrow$  will mean zero:  $1 \leftarrow = 0$ .)

(34)  $22 \leftarrow \leftarrow = (22 \leftarrow) \leftarrow = 21 \leftarrow$

This says that the symbol  $22 \leftarrow \leftarrow$  names the same thing that  $21 \leftarrow$  names. Unfortunately, if we have not extended our system, the symbol  $21 \leftarrow$  is meaningless—it does not name anything. Hence, neither does  $22 \leftarrow \leftarrow$ .

(If you choose, say, the "rubber-stamp" extension,  $21 \leftarrow = 30$ , and so  $22 \leftarrow \leftarrow = 30$ . If, instead, you chose the "barber-pole" extension, or the "arithmetical" extension, then  $22 \leftarrow \leftarrow = 20$ .)

(35) This depends upon your class.

(36) This depends upon your class. (My own choice is the "arithmetic" or "barber-pole" extension.)

We now build the "simple" or "basic" part of the system of exponents.

We can often make these problems simpler, and easier to think about, if we introduce the system that mathematicians call **exponents**. Here is how it works:

$$\begin{aligned}2^2 &= 2 \times 2 = 4 \\2^3 &= 2 \times 2 \times 2 = 8 \\2^4 &= 2 \times 2 \times 2 \times 2 = 16 \\2^5 &= 2 \times 2 \times 2 \times 2 \times 2 = 32 \\2^6 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64\end{aligned}$$

$$\begin{aligned}3^2 &= 3 \times 3 = 9 \\3^3 &= 3 \times 3 \times 3 = 27 \\3^4 &= 3 \times 3 \times 3 \times 3 = 81\end{aligned}$$

$$\begin{aligned}5^2 &= 5 \times 5 = 25 \\5^3 &= 5 \times 5 \times 5 = 125 \\5^4 &= 5 \times 5 \times 5 \times 5 = 625\end{aligned}$$

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$$\begin{aligned}10^2 &= 10 \times 10 = 100 \\10^3 &= 10 \times 10 \times 10 = 1000 \\10^4 &= 10 \times 10 \times 10 \times 10 = 10,000 \\10^5 &= 10 \times 10 \times 10 \times 10 \times 10 = 100,000\end{aligned}$$

Can you find simpler names for each of these numbers?

(37)  $4^2$

(37)  $4^2 = 4 \times 4 = 16$

(38)  $7^2$

(38)  $7^2 = 7 \times 7 = 49$

(39)  $1^5$

(39)  $1^5 = 1 \times 1 \times 1 \times 1 \times 1 = 1$

(40)  $2^{10}$

$$\begin{aligned}(40) 2^{10} &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\&= (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \times (2 \times 2) \\&= 16 \times 16 \times 4 \\&= 256 \times 4 \\&= 1024\end{aligned}$$

(41)  $100^2$

(41)  $100^2 = 100 \times 100 = 10,000$

(42)  $11^2$

(42)  $11^2 = 11 \times 11 = 121$

(43)  $(-1)^2$

(43)  $(-1)^2 = -1 \times -1 = +1$

(44)  $(-1)^3$

$$\begin{aligned}(44) (-1)^3 &= -1 \times -1 \times -1 \\&= (-1 \times -1) \times -1 \\&= +1 \times -1 \\&= -1\end{aligned}$$

(45)  $(-1)^4$

$$\begin{aligned} (45) \quad (-1)^4 &= -1 \times -1 \times -1 \times -1 \\ &= (-1 \times -1) \times (-1 \times -1) \\ &= +1 \times +1 \\ &= +1 \end{aligned}$$

(46)  $(-1)^5$

$$\begin{aligned} (46) \quad (-1)^5 &= -1 \times -1 \times -1 \times -1 \times -1 \\ &= (-1 \times -1 \times -1 \times -1) \times -1 \\ &= +1 \times -1 \\ &= -1 \end{aligned}$$

(47)  $1^{1965}$

(47)  $1^{1965} = 1$

(48)  $(-1)^{1965}$

(48)  $(-1)^{1965} = -1$

Obviously, we don't want to tackle this problem by brute force. What we need here is an *idea*! Let's group according to ALM, much as we did in question 46. Thus  $(-1)^{1965}$  will mean 1965 factors of negative one. If we group the first four together, we will have

$$(-1)^{1965} = (-1)^4 \times (-1)^{1961},$$

since there will be 1961 factors left. Now, we saw in question 45 that  $(-1)^4 = +1$ ; hence,  $(-1)^{1965} = +1 \times (-1)^{1961} = (-1)^{1961}$ . If we group four factors of  $-1$ , again, we have

$$(-1)^{1965} = (-1)^4 \times (-1)^{1957},$$

since  $1961 - 4 = 1957$ . Hence,  $(-1)^{1965} = +1 \times (-1)^{1957} = (-1)^{1957}$ . We can "remove" four more factors of  $-1$ , by this same process of grouping according to ALM, to get

$$(-1)^{1965} = (-1)^{1953}$$

Where will this process end? Obviously, when we can't remove any further groups of four factors of  $-1$ . Now,

$$\begin{aligned} 1953 &= 4 + 1949 \\ &= 4 + 4 + 1945 \\ &= 4 + 4 + 4 + 1941 \\ &= 4 + 4 + 4 + 4 + 1937, \end{aligned}$$

and so on. If we remove *all possible* groups of four, we are left with a "remainder" of 1:

$$\begin{array}{r} 484 \text{ R}1 \\ 4 \overline{)1937} \end{array}$$

That is, we shall ultimately get down to

$$\begin{aligned} (-1)^{1965} &= (-1)^4 \times -1 \\ &= +1 \times -1 \\ &= -1. \end{aligned}$$

Thus the final answer is

$$(-1)^{1965} = -1.$$

Once we see how this works, we can save a little time in the future. We might merely have done this:

$$(-1)^{1965} = ?$$

$$\begin{array}{r} \phantom{0} 491 \text{ R1} \\ 4 \overline{) 1965} \end{array}$$

From this we see at once that, after we have removed 491 groups of 4, we shall be left with a "remainder" of *one single factor of -1*. Since each group of 4 meant merely multiplication by +1, the result is, finally, -1. (We could, instead, have removed groups of two factors, since  $-1 \times -1 = +1$ .)

(49)  $(-1)^{1066}$

(49) **Removing groups of 4,**

$$\begin{array}{r} \phantom{0} 266 \text{ R2} \\ 4 \overline{) 1066} \end{array}$$

so, after we remove 266 groups of 4, we shall be left with a "remainder" of 2 factors of -1:

$$\begin{aligned} (-1)^{1066} &= [(-1)^4]^{266} \times (-1)^2 \\ &= (+1)^{266} \times (-1)^2 \\ &= +1 \times (-1)^2 \\ &= -1 \times -1 \\ &= +1 \end{aligned}$$

(Note that we might, instead, have removed groups of 2 factors of -1.)

(50)  $(-1)^{200}$

(50)  $(-1)^{200} = +1$

(51)  $(-1)^{201}$

$$\begin{aligned} (51) \quad (-1)^{201} &= (-1)^{200} \times -1 \\ &= +1 \times -1 \\ &= -1 \end{aligned}$$

(52)  $10^{100}$

$$\begin{aligned} (52) \quad 10^{100} &= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &\quad \times 10 \times 10 \times 10 \times 10 \\ &= 10,000,000,000,000,000,000,000,000,000,000,000, \\ &\quad 000,000,000,000,000,000,000,000,000,00 \\ &\quad 0,000,000,000,000,000,000,000,000,0 \\ &\quad 00,000,000 \end{aligned}$$

Which statements are true and which are false?

(53)  $2^{10} < 10^3$

(53) **False.** Since  $2^{10} = 1024$  and  $10^3 = 1000$ , the statement  $2^{10} < 10^3$  is really the same as (using PN)  $1024 < 1000$ , which is false.

(54)  $2^{10} < 10^2$

(54) **False.**  $2^{10} < 10^2$  is really the same as  $1024 < 100$ .

(55)  $2^{10} < 10^4$

(55) **True.**  $2^{10} < 10^4$  is really the same statement as  $1024 < 10,000$ .

(56)  $5^4 < 4^5$

(56) **True.**  $5^4 < 4^5$  is really the same statement as  $625 < 1024$ .

(57)  $3^4 < 4^3$

(57) **False.**  $3^4 < 4^3$  is really the same statement as  $81 < 64$ .

Can you find the truth set for each open sentence?

(Let's agree to use only positive integers.)

(58)  $2^6 = \square^3$

$$\begin{aligned} (58) \quad 2^6 &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= (2 \times 2) \times (2 \times 2) \times (2 \times 2) \\ &= 4 \times 4 \times 4 \\ &= 4^3 \end{aligned}$$

Consequently, the open sentence  $2^6 = \square^3$  has the truth set  $\{4\}$ . Notice that we could also write  $2^6 = (2^2)^3$ .

(59)  $3^4 = 9^2$

$$\begin{aligned} (59) \quad 3^4 &= 3 \times 3 \times 3 \times 3 \\ &= (3 \times 3) \times (3 \times 3) \\ &= 9 \times 9 \\ &= 9^2 \end{aligned}$$

Hence, the open sentence  $3^4 = 9^2$  has the truth set  $\{2\}$ . Notice that we could also write  $3^4 = (3^2)^2$ .

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(60)  $5^{10} = 25^5$

$$\begin{aligned} (60) \quad 5^{10} &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= (5 \times 5) \times (5 \times 5) \times (5 \times 5) \times (5 \times 5) \times (5 \times 5) \\ &= 25 \times 25 \times 25 \times 25 \times 25 \\ &= 25^5 \end{aligned}$$

Hence, the open sentence  $5^{10} = 25^5$  has the truth set  $\{5\}$ . We could also write  $5^{10} = (5^2)^5$ . Does this suggest anything?

(61)  $2^3 \times 2^4 = 2^7$

$$\begin{aligned} (61) \quad 2^3 \times 2^4 &= (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^7 \end{aligned}$$

Thus, the truth set is  $\{7\}$ . It shouldn't be surprising that 3 factors (of 2) plus 4 factors, turn out to be 7 factors.

(62)  $2^4 \times 2^3 = 2^7$

(62)  $\{8\}$

(63)  $3^2 \times 3^3 = 3^5$

(63)  $\{5\}$

(64)  $3^3 \times \square^2 = 3^7$

(64)  $\{3\}$

(65)  $2^3 \times 5^3 = \square^3$

$$\begin{aligned} (65) \quad (2 \times 2 \times 2) \times (5 \times 5 \times 5) &= (2 \times 5) \times (2 \times 5) \times (2 \times 5) \\ &= (2 \times 5)^3 \\ &= 10^3 \end{aligned}$$

Hence, the open sentence  $2^3 \times 5^3 = \square^3$  has the truth set  $\{10\}$ .

(66)  $2^2 \times 3^2 = \square^2$

(67)  $(-1)^{201} \times (-1)^{201} = \square$

(68)  $2^9 < 10^0 < 2^{10}$

(69)  $25 < 7^0 < 50$

(70)  $2^6 < 5^0 < 2^7$

$$\begin{aligned}
 (66) \quad (2 \times 2) \times (3 \times 3) &= (2 \times 3) \times (2 \times 3) \\
 &= (2 \times 3)^2 \\
 &= 6^2
 \end{aligned}$$

Hence the open sentence  $2^2 \times 3^2 = \square^2$  has the truth set  $\{6\}$ .

(67)  $\{1\}$

(68) We might attack this problem as follows:

$$\begin{aligned}
 10^2 &= 100 \\
 10^3 &= 1000 \\
 10^4 &= 10,000 \\
 2^9 &= 512 \\
 2^{10} &= 1024
 \end{aligned}$$

Our problem can be written as

$$512 < 10^0 < 1024.$$

The only power of 10 which falls between 512 and 1024 is, evidently, 1000.

That is,

$$512 < 1000 < 1024$$

or

$$512 < 10^3 < 1024.$$

Consequently, the open sentence  $512 < 10^0 < 1024$  has the truth set  $\{3\}$ .

(69) We might attack this problem this way:

$$\begin{aligned}
 7^2 &= 49 \\
 7^3 &= 343
 \end{aligned}$$

Clearly, then,  $25 < 49 < 50$ . In other words, the open sentence  $25 < 7^0 < 50$  has the truth set  $\{2\}$ .

(70)  $2^6 = 64$

$2^7 = 128$

So the open sentence

$$2^6 < 5^0 < 2^7$$

can be written as

$$64 < 5^0 < 128.$$

Now let's see which powers of 5 might be candidates:

$$\begin{aligned}
 5^2 &= 25 \\
 5^3 &= 125 \\
 5^4 &= 625 \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

Consequently, the open sentence  $64 < 5^0 < 128$  has the truth set  $\{3\}$ .

(71)  $10 < 2^0 < 10^2$

(71)  $10 < 2^0 < 100$

Which powers of 2 might be candidates? Let's look at a few:

$$\begin{aligned} 2^2 &= 4 \\ 2^3 &= 8 \\ 2^4 &= 16 \\ 2^5 &= 32 \\ 2^6 &= 64 \\ 2^7 &= 128 \\ &\vdots \\ &\vdots \end{aligned}$$

So, evidently, the open sentence  $10 < 2^n < 100$  has the truth set  $\{4, 5, 6\}$ .

(72)  $10 < 3^0 < 10^2$

(72)  $10 < 3^0 < 100$

Let's look at some powers of 3:

$$\begin{aligned} 3^2 &= 9 \\ 3^3 &= 27 \\ 3^4 &= 81 \\ 3^5 &= 243 \\ &\vdots \\ &\vdots \end{aligned}$$

Evidently, the open sentence  $10 < 3^n < 100$  has the truth set  $\{3, 4\}$ .

(73)  $10^2 < 5^0 < 10^3$

(73)  $100 < 5^0 < 1000$

Let's look at some powers of 5:

$$\begin{aligned} 5^2 &= 25 \\ 5^3 &= 125 \\ 5^4 &= 625 \\ 5^5 &= 3125 \\ &\vdots \\ &\vdots \end{aligned}$$

The open sentence  $100 < 5^n < 1000$  has the truth set  $\{3, 4\}$ .

We have now developed some familiarity with the simple, "basic" part of our mathematical system, namely, positive integer exponents. We now begin to move toward the unfinished "frontier" of our system.

If we use only positive integers as replacements for the variables, which of these are identities?

(74)  $\square^2 \times \square^3 = \square^6$

(74) Not an identity. For example, try

UV:  $2 \rightarrow \square$

$$(2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

which is false.

(75)  $\square^2 \times \square^3 = \square^5$

(75) An identity.

(76)  $(\square^2)^3 = \square^6$

(76) An identity.

(77)  $(\square^2)^3 = \square^5$

(77) Not an identity.

(78)  $p^a \times q^b = (p \times q)^{a \times b}$

(79)  $p^a \times p^b = p^{a \times b}$

(80)  $p^a \times p^b = p^{a+b}$

(81)  $p^a \times q^a = (p \times q)^{a \times a}$

(82)  $p^a \times q^a = (p \times q)^a$

(83)  $p^a \div p^b = p^{a-b}$

(78) **Not an identity. For example, try**

$$\begin{array}{l} \text{UV: } 2 \rightarrow p \\ \quad 3 \rightarrow q \\ \quad 4 \rightarrow a \\ \quad 2 \rightarrow b \end{array}$$

to get  $2^4 \times 3^2 = (6)^8$ , which is false.(79) **Not an identity.**(80) **An identity.**(81) **Not an identity.**(82) **An identity.**(83) **Here we encounter the frontier:  $p^a \div p^b = p^{a-b}$  is an identity, for positive integers, provided  $a > b$ .**

Let's try it out with some numbers.

$$\begin{array}{l} \text{UV: } 2 \rightarrow p \\ \quad 6 \rightarrow a \\ \quad 3 \rightarrow b \end{array}$$

We get  $2^6 \div 2^3 = 2^3$ . Is this statement true? Can we see "why" it is true?

$$\begin{aligned} \frac{2^6}{2^3} &= \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = \frac{(2 \times 2 \times 2)}{2 \times 2 \times 2} \times (2 \times 2 \times 2) \\ &= 1 \times (2 \times 2 \times 2) \\ &= 2^3 \end{aligned}$$

That worked out smoothly enough. But, of course, we satisfied the condition  $a > b$ .Suppose, instead, we had  $a = b$ . What then? Let's try it.

$$\begin{array}{l} \text{UV: } 2 \rightarrow p \\ \quad 5 \rightarrow a \\ \quad 5 \rightarrow b \end{array}$$

We get  $2^5 \div 2^5 = 2^0$ . Now—what on earth can we mean by the symbol  $2^0$ ? Well,

$$\frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = 1.$$

Consequently, if we want to retain the identity  $p^a \div p^b = p^{a-b}$  for the case where  $a = b$ , we must agree to use  $2^0$  as a name for the number 1,  $2^0 = 1$ .To summarize, for positive integers  $a$ ,  $b$ , and  $p$ , the open sentence  $p^a \div p^b = p^{a-b}$  is an identity, provided  $a > b$ . If  $a = b$ , we suddenly hit the frontier of our system; we get the symbol  $2^0$ , or, in general,  $p^0$ , which has no meaning.We can extend our system by giving a meaning to the symbol  $p^0$ , where  $p$  is a positive integer.If we want the open sentence  $p^a \div p^b = p^{a-b}$  to continue to be an identity, even for our extended system, then there is only one meaning that we can give to  $p^0$ . We must give it the meaning

$$p^0 = 1.$$



(Remember that here,  $p$  is some positive integer.) We have now extended our system of exponents to allow exponents 0, 1, 2, 3, ..., that is, positive integers for exponents, and also zero as an exponent.

Can we extend our system still further? Well, as a matter of fact, we more or less *need* to do so. For, the open sentence  $p^a \div p^b = p^{a-b}$  is an identity *provided*  $a \geq b$ . But suppose  $a < b$ ; what happens then? We get, for example, this sort of situation:

$$\begin{array}{l} UV: 2 \rightarrow p \\ 3 \rightarrow a \\ 5 \rightarrow b \end{array}$$

from which we get  $2^3 \div 2^5 = 2^{-2}$  since  $3 - 5 = -2$ . Now, what on earth can we possibly mean by the symbol  $2^{-2}$ ? Well,

$$\begin{aligned} 2^3 \div 2^5 &= \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} \\ &= \frac{2 \times 2 \times 2}{2 \times 2 \times 2} \times \frac{1}{2 \times 2} \\ &= 1 \times \frac{1}{2^2} \\ &= \frac{1}{2^2} \end{aligned}$$

Hence, if we want the open sentence  $p^a \div p^b = p^{a-b}$  to be an identity, even for  $a < b$ , then we must give  $2^{-2}$  the meaning

$$2^{-2} = \frac{1}{2^2}.$$

In general, we *can* extend our system to allow for negative integers as exponents. When we make this extension, if we want the open sentence  $p^a \div p^b = p^{a-b}$  to *continue* to be an identity, *even* for  $a < b$ , then there is only one meaning we can give to the symbol  $p^{-2}$ . We *must* give it the meaning

$$p^{-2} = \frac{1}{p^2}.$$

(Similarly,  $p^{-3}$  must be given the meaning

$$p^{-3} = \frac{1}{p^3},$$

the symbol  $p^{-4}$  must be given the meaning

$$p^{-4} = \frac{1}{p^4},$$

and so on. Remember,  $p$  is some positive integer.)

We have now extended our system so that any integer — positive, negative, or zero — can be used as an exponent.

(84) Do you see any need to extend our system of exponents?

(85) Thus far, we have *been* using the set of positive integers as the replacement set for our variables. What other set might we use, instead of merely positive integers?

(84) **This has just been discussed above.**

(85) **We might include zero, negative integers, and fractions as exponents. (Actually, one *could* even go further, and would do so in advanced mathematics.)**

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(86) Do you see any reason to extend our system of exponents?

Can your extended system cope with these problems?  
Can you find simpler names for any of these numbers?

(87)  $3^1$

(88)  $3^0$

(89)  $2^0$

(90)  $2^{-1}$

(91)  $3^{-1}$

(92)  $2^{-5}$

(93)  $10^0$

(94)  $10^{-1}$

(95)  $10^{-2}$

(96)  $10^{-3}$

(97)  $10^{-10}$

(98)  $10^5 \div 10^7 = \square$

(99)  $2^3 \div 2^6 = \square$

(100) Mary says that if there really is such a thing as  $9^{\frac{1}{2}}$ , then

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9.$$

Is Mary right? Do you think there really is such a thing as  $9^{\frac{1}{2}}$ ?

(86) This has already been discussed (although your class may wish to carry the discussion further).

(87)  $3^1 = 3$

We have not said much about this; we leave it to you and your class. It may not even require discussion.

(88)  $3^0 = 1$

(89)  $2^0 = 1$

(90)  $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

(91)  $3^{-1} = \frac{1}{3}$

(92)  $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

(93)  $10^0 = 1$

(94)  $10^{-1} = \frac{1}{10} = 0.1$

(95)  $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$

(96)  $10^{-3} = \frac{1}{10^3} = 0.001$

(97)  $10^{-10} = \frac{1}{10^{10}} = 0.0000000001$

(98)  $10^5 \div 10^7 = 10^{-2} = \frac{1}{10^2} = 0.01$

That is, the open sentence  $10^5 \div 10^7 = \square$  has the truth set  $\{10^{-2}\}$ .

(99)  $\{\frac{1}{2}\}$

(100) **Yes, Mary is right.**

Here is a new frontier! What on earth can we mean by the symbol  $9^{\frac{1}{2}}$ ?

Can we extend our mathematical system to cope with problems like this? Well,  $p^a \times p^b = p^{a+b}$ .

Let's use UV this way:

$$\begin{aligned} UV: \frac{1}{2} &\rightarrow a \\ \frac{1}{2} &\rightarrow b \\ 9 &\rightarrow p \end{aligned}$$

Thus we get  $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^1$ , or  $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9$ . Hence, we should extend our system by using the symbol  $9^{\frac{1}{2}}$  to refer to some element of the truth set of the open sentence  $\square \times \square = 9$ . Now, this truth set is  $\{+3, -3\}$ .

Which meaning shall we give to our "meaningless" symbol  $9^{\frac{1}{2}}$ ? Shall we say  $9^{\frac{1}{2}} = +3$ , or shall we say  $9^{\frac{1}{2}} = -3$ ? Either might be possible, but mathematicians have chosen to give  $9^{\frac{1}{2}}$  the meaning

$$9^{\frac{1}{2}} = 3.$$

In other words,  $9^{\frac{1}{2}} = \sqrt{9}$ , if we remember that the symbol  $\sqrt{\quad}$  always refers to the nonnegative square root (if there is one).

We have now extended our system to allow positive integers, zero, negative integers, and fractions (either positive or negative) to serve as exponents. That is, we can now use any *rational number* as an exponent in our new extended system!

Can your extended system cope with these problems?  
Can you find simpler names for any of these numbers?

(101)  $100^{\frac{1}{2}}$

(102)  $27^{\frac{1}{3}}$

(103)  $1000^{\frac{1}{3}}$

(104)  $10,000^{\frac{1}{2}}$

(105)  $1024^{\frac{1}{5}}$

(101)  $100^{\frac{1}{2}} = \sqrt{100} = 10$

(102)  $27^{\frac{1}{3}}$ —what can this mean? That's easy:

$$27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}} = 27, \text{ so } 27^{\frac{1}{3}} = \sqrt[3]{27} = 3.$$

(103)  $1000^{\frac{1}{3}} = \sqrt[3]{1000} = 10$

(104)  $10,000^{\frac{1}{2}} = \sqrt{10,000} = 100$

(105)  $1024^{\frac{1}{5}}$ —what can this mean? Let's try this:

$$1024^{\frac{1}{5}} \times 1024^{\frac{1}{5}} \times 1024^{\frac{1}{5}} \times 1024^{\frac{1}{5}} \times 1024^{\frac{1}{5}} = 1024$$

Now,

$$2^{10} = 1024$$

$$2^2 \times 2^2 \times 2^2 \times 2^2 \times 2^2 = 1024$$

and so

$$1024^{\frac{1}{5}} = 2^2 = 4.$$

Using the symbol

$$\sqrt[5]{\quad}$$

to denote what is called the "fifth root" of a number, we can write

$$1024^{\frac{1}{5}} = \sqrt[5]{1024} = 4.$$

We could have tackled this problem in many different ways. Here is a nice approach. In the old system,  $(p^r)^s = p^{rs}$  has been an identity. If we want it to continue to be an identity in the new *extended system*, we must have:

$$\text{UV: } 2 \rightarrow p$$

$$10 \rightarrow r$$

$$\frac{1}{5} \rightarrow s$$

If we use rational numbers as replacements for the variables  $a$  and  $b$ , which of the following are identities?

(106)  $\square^a \times \square^b = \square^{a \times b}$

(106) Not an identity.

(107)  $\square^a \times \square^b = \square^{a+b}$

(107) An identity.

(108)  $p^a \times q^a = (p \times q)^a$

(108) An identity.

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(109)  $\square^a \div \square^b = \square^{a-b}$

(109) An identity.

(110)  $(\square^a)^b = \square^{a+b}$

(110) Not an identity.

(111)  $(\square^a)^b = \square^{a \times b}$

(111) An identity.

(112) Can you describe how you extended the original system of exponents? Why did you do it the way that you did?

(112) This discussion will depend upon your class.



## GUESSING FUNCTIONS<sup>23</sup>

Before teaching this lesson, you may want to view the Madison Project film entitled "Guessing Functions." This film shows a class of culturally deprived urban children in grades 6 and 7. (At least, their teachers assure me that these children are culturally deprived. You could never tell it by looking at them or by working with them on mathematical problems.)

What do mathematicians mean by a *function*? Well, let's first observe what we might call a *rule*, and what we might call a *formula*.

Here is a *rule*: Whatever number you tell me, I'll double it, then add six, and tell you the answer. Here is a *formula*, which is a way to use *variables* in writing a *rule*: Let  $\square$  represent the number you tell me and let  $\triangle$  represent the number I answer back. Then the preceding *rule* can be written down as the *formula*

$$(\square \times 2) + 6 = \triangle.$$

Here is a different rule: Whatever number you tell me, I'll add three to it, double the result, and then tell you the answer. The formula for this second rule is

$$(\square + 3) \times 2 = \triangle.$$

Are these two rules (or formulas) different, or not? Well, they certainly can be distinguished from one another; one looks like this:

$$(\square \times 2) + 6 = \triangle.$$

And the other looks like this:

$$(\square + 3) \times 2 = \triangle.$$

So we have to admit that they *are* different.

However, if we make up a table using the rule  $(\square \times 2) + 6 = \triangle$ , we get:

$\square$	$\triangle$
0	6
1	8
2	10
3	12
⋮	⋮
⋮	⋮
⋮	⋮

<sup>23</sup>This topic was suggested by Professor W. Warwick Sawyer of Wesleyan University, Middletown, Connecticut. Professor Sawyer has been one of the international leaders in mathematics curriculum reform for many years.

If we make up a table using the rule  $(\square + 3) \times 2 = \triangle$ , we get:

$\square$	$\triangle$
0	6
1	8
2	10
3	12
$\vdots$	$\vdots$
$\vdots$	$\vdots$

so we get the same table from either rule.\*

Mathematicians express this by invoking a higher abstraction, known as a *function*.<sup>1</sup> We say that both formulas represent the same function. What, then, is a *function*? Well, it is more abstract. It refers to the "commonality" or "sameness" shared by both of our formulas. (In a similar sense, the *number* two refers to the commonality shared by two fingers, two pebbles, two trees, two houses, two boys, two legs, two letters of the alphabet, two avocados, etc.)

We can make up a *set* or *collection* or *class* of formulas, by agreeing to put a formula into the set if it produces the same table that  $(\square \times 2) + 6 = \triangle$  does; otherwise we agree to leave it out of the set.

We have thus made up a set of all *formulas* which produce the same table that  $(\square \times 2) + 6 = \triangle$  does. Mathematicians call this set an *equivalence class of formulas*. Here is part of what it looks like:

$$\begin{aligned} &\{(\square \times 2) + 6 = \triangle, (\square + 3) \times 2 = \triangle, \\ &2 \times (\square + 3) = \triangle, 2 \times (3 + \square) = \triangle, \\ &6 + (\square \times 2) = \triangle, \frac{1}{2} \times [12 + (\square \times 4)] = \triangle, \dots\} \end{aligned}$$

Consequently, we could think of a *function* as an *equivalence class of formulas that produce the same table*. We would then say that any formula in the set "represents" the function.

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\*Of course, the " $\triangle$ " numbers must always come out the same, since

$$(\square \times 2) + 6 = (\square + 3) \times 2$$

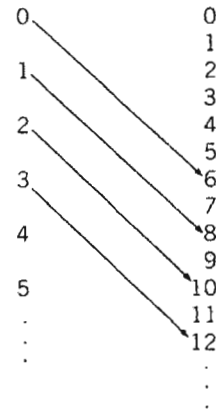
is an *identity*.

<sup>1</sup>Please notice the danger here. As so often happens, mathematicians have chosen to use an everyday word, but not with its everyday meaning. They have given it a special new "mathematical" meaning. When used in mathematics, such words must not be given their ordinary meaning. This common practice of using ordinary words with special meanings will give us great difficulty if we are not on the alert. "Set" in mathematics does not mean what it does in everyday life (as in "the gelatin will set" or "mental set," or "set them down over there"). Here are some more words with special meanings in mathematics: function, radical, ring, group, element, rational, irrational, real, complex, imaginary, opposite, exponent, power, prime, open, closed, integrate, differentiate, limit, bound, infinity ... and lots more. In mathematics, for example, the opposite of the opposite need not be the original element. One can become very confused if one fails to separate the mathematical meanings of these words from their everyday meanings.

There are other ways to explain what we mean by a function. One valuable way (that takes some getting used to, at first, but is often helpful intuitively) is to say that

a function is a *mapping*

which means, roughly, what we shall try to suggest by this picture:



If we pursue this notion, we shall be led to study what we mean by *mappings*. This is an extremely important branch of mathematics, but is too large to be dealt with briefly. (We shall study *mappings* further in Chapter 36.)

One approach has been popular in recent years, and goes as follows. Since somehow the *sameness* of the "different" formulas is revealed primarily by the table

□	△
0	6
1	8
2	10
3	12
⋮	⋮
⋮	⋮

we shall focus on this *table*. Now, what is a table? It is a set of ordered pairs of numbers, such as

$$\{(0, 6), (1, 8), (2, 10), (3, 12), \dots\}.$$

If we say zero, we get the answer six; if we say one, we get the answer eight; and so on. The *order* lets us distinguish the number that we say (which always appears as the first number of each pair) from the number that we get as an *answer* (which always appears as the second number of the pair).

Now, for a rule to work smoothly, we want the function to be *single valued*; that is, whenever we say a number, we want to get a *definite answer*—not two or three answers. Not all functions are single-valued, although recent authors of precollege textbooks have tended to use "function" to mean the same thing as "single-valued function." It is worth emphasizing that *not every problem has a single answer*.

Perhaps because matters seem simplest when each question has a single answer, we have apparently overemphasized this case

with young children. As a result, our students often acquire the quite erroneous idea that every question must have exactly one answer. Clearly, this is false. If we say

F.D.R. was President in the year

we can answer 1933, 1934, 1935, or—in fact—any year until 1945, inclusive. Thus, the *truth set* for the open sentence

F.D.R. was President in the year

is the set

$$\{1933, 1934, \dots, 1945\}.$$

Again, if we use only integers, the open sentence

$$3 < \square + 1 \leq 8$$

has the truth set

$$\{3, 4, 5, 6, 7\},$$

so this “question” has five different “answers.”

If we ask, “What number, when multiplied by zero, will yield 37?” the answer is “no number will!” The open sentence

$$0 \times \square = 37$$

has the empty set for its truth set. There is no “answer” to this question, except to say that there is no answer—which is, after all, a kind of “answer.”

(Obviously, the language becomes considerably clearer and simpler when, instead of speaking of “answers,” we speak of *truth sets*, and how many elements there are in the truth sets.)

The important point, here, is that in the present chapter we shall be working with single-valued functions. But we must not let the simplicity of *this* situation mislead us in the future. *Not every important question has one single simple answer. Some have many answers, some have none; and sometimes the “answer” is extremely complicated, and may consist of many parts.*

If the set of ordered pairs looked like the set

$$\{(0, 5), (0, 3), (1, 6), (1, 4), (2, 7), \dots\},$$

then we would *not* get one definite answer to each number that we started with. If we said, for example, zero we *might* get the answer five or we *might* get the answer three. If we said one, we *might* get the answer six or we *might* get the answer four. What must a table look like, in order to avoid this kind of ambiguity?

Evidently, if 1 appears once as a “starting” number,

$$\begin{array}{c|c} \square & \triangle \\ \hline 1 & \end{array}$$

it can be paired up with any “answering number” (say, 12),

$$\begin{array}{c|c} \square & \triangle \\ \hline 1 & 12 \end{array}$$



but thereafter 1 can never appear in the □ column paired up with any number other than 12 in the △ column. For example, we must not have:

□	△
1	12
2	13
3	14
1	8

If we did, when we said 1 we might get *either* the answer 12 *or* the answer 8 – the function would *not* be “single-valued.”

Using letters to denote variables, we can say the same thing this way: In the table

□	△
a	r
b	s

if  $a = b$ , then we *must* have  $r = s$ .

Most recent authors, consequently, have defined a *function* as a set of ordered pairs, where two different pairs never have the same first element.

It might be best to leave it for you to decide how you want to discuss the concept of *function* with your students. The main idea is that, although

$$(\square \times 2) + 6 = \triangle$$

and

$$(\square + 3) \times 2 = \triangle$$

are different formulas, they produce the same table:

□	△
0	6
1	8
2	10
⋮	⋮
⋮	⋮

Hence, we say *they both represent the same function*.

The function, then, refers to how we have paired up numbers in our table, and not to the rule by which we did it. Different rules, as we have seen, can result in pairing up numbers in the same way.

The study of functions is one of the most important topics in all of mathematics. Moreover, it's easy and fun. One can use the idea of this chapter quite effectively with elementary school children or, for that matter, with high school and college students.

In these preliminary remarks we have tried to say what a function is – and that is a somewhat cumbersome matter. But in the rest of this chapter, *and in our work with children*, we shall merely be “making up rules” and “guessing rules,” and this will be easy and enjoyable.





## CHAPTER 25

## Guessing Functions

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(1) Ranny made up a rule and Alec tried to guess what it was.

When Alec told Ranny "zero," Ranny answered "three."

When Alec said "one," Ranny answered "five."

When Alec said "two," Ranny said "seven."

$\square$	$\triangle$
0	3
1	5
2	7

Do you know what rule Ranny made up?

## ANSWERS AND COMMENTS

(1) Telling somebody "zero" is usually revealing. Since Ranny answered "three," we can reasonably hope that Ranny's rule is something like the following:

$$(\_ \times \square) + 3 = \triangle$$

$$(\square \times \square) + 3 = \triangle$$

$$3 - (\_ \times \square) = \triangle$$

$$3 - (\_ ) \times (\square \times \square) = \triangle$$

In the next chapter we shall try to be somewhat more systematic in our approach to "guessing functions." The present chapter is concerned — from the student's point of view — with a kind of "background experience" or *readiness building*. Hence we would *not* tell the students, at this stage, how to go about "guessing." But it may be well for the teacher to know some systematic methods, *even though we won't tell them to the children just yet!*

In the present case, let us look for a moment at the possible *forms* of the answer. We might have just a "constant," that ignores the number Alec says; that is, Ranny always answers the same number, regardless of what number Alec tells him. In that case, since Ranny has already answered "three," the table would look like this:

$\square$	$\triangle$
0	3
1	3
2	3
3	3
4	3
5	3
⋮	⋮
⋮	⋮

And the "formula" would be:

$$3 = \triangle.$$

However, a quick glance at the *actual* table

$\square$	$\triangle$
0	3
1	5
2	7
⋮	⋮
⋮	⋮

shows us that this is *not* the kind of rule that Ranny is using.

Well, perhaps the next simplest rule would be one where Ranny took Alec's number, multiplied by some definite "fixed" number, added some definite "fixed" number, and told us the answer. The formula for a rule of this kind would look like this:

$$(\square \times 3) + 5 = \triangle$$

or perhaps

$$(\square \times 1) + 2 = \triangle$$

or perhaps

$$(\square \times 2) + 3 = \triangle$$

or whatever. In general, the "form" of the formula would be

$$(\square \times \text{---}) + \text{---} = \triangle$$

↗
↖

Some definite "fixed" number that Ranny chose in advance      Some definite "fixed" number that Ranny chose in advance.

Now, how can we tell if Ranny's rule is of this form? Actually, *there is a systematic way to tell!* We arrange the  $\square$  numbers in proper sequential order (1, 2, 3, 4, 5, ..., etc.) and then we look to see if the *differences* between successive  $\triangle$  numbers are the same. Perhaps it is easier to *show* what this means than to try to say what it means:

$\square$	$\triangle$	Differences
0	3	
1	5	2
2	7	2

Now, *if this pattern continues*, for example, if Ranny's rule works like the following,

$\square$	$\triangle$
0	3
1	5
2	7
3	9
4	11
5	13
⋮	⋮

then Ranny is using a rule of the form

$$(\square \times \text{---}) + \text{---} = \triangle$$

↑
↑

Some definite number      Some definite number

On the other hand, if this pattern (of the differences all being equal to 2) does *not* continue, then Ranny is *not* using a rule of this type.

Mathematicians have a word to describe rules of this type – they are called *linear*. Thus, each of the following rules is linear:

$$(\square \times 1) + 10 = \triangle$$

$$(\square \times 2) + 4 = \triangle$$

$$(\square \times 5) + 3 = \triangle$$

$$(\square \times 1) + 8 = \triangle$$

We can use letters to represent “definite fixed numbers chosen before the guessing begins,” and write such rules this way:

$$(\square \times a) + b = \triangle.$$

Notice what their tables look like:

$\square$	$\triangle$
0	10
1	9
2	8
3	7
4	6
⋮	⋮
⋮	⋮

Table corresponding to  $(\square \times 1) + 10 = \triangle$   
(all differences between successive  $\triangle$  numbers equal 1)

$\square$	$\triangle$
0	4
1	6
2	8
3	10
4	12
⋮	⋮
⋮	⋮

Table corresponding to  $(\square \times 2) + 4 = \triangle$   
(all differences between successive  $\triangle$  numbers equal 2)

$\square$	$\triangle$
0	3
1	8
2	13
3	18
4	23
⋮	⋮
⋮	⋮

Table corresponding to  $(\square \times 5) + 3 = \triangle$   
(all differences between successive  $\triangle$  numbers equal 5)

$\square$	$\triangle$
0	8
1	7
2	6
3	5
4	4
⋮	⋮
⋮	⋮

Table corresponding to  $(\square \times 1) + 8 = \triangle$   
(all differences between successive  $\triangle$  numbers equal 1)

All of the preceding rules are *linear*. Let us look now at some rules which are *not linear*:

$\square$	$\triangle$
0	3
1	7
2	10
3	12
4	13
⋮	⋮
⋮	⋮

How do we *know* this rule is nonlinear? We need only look at the differences between successive  $\triangle$  numbers:

$\square$	$\triangle$	Differences
0	3	4
1	7	
2	10	3
3	12	2
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

Since these differences are *not* all the same, the "rule" (or "formula" or "function") is *not linear*. Therefore, it *cannot be written in the form*

$$(\square \times a) + b = \triangle,$$

with some definite numbers for *a* and *b*. It would be a waste of time, *in this case*, to try to work with the form

$$(\square \times a) + b = \triangle.$$

As a second example of a nonlinear rule, if we make a table for the rule  $(\square \times \square) + 1 = \triangle$ , we get:

$\square$	$\triangle$	Differences
0	1	1
1	2	
2	5	3
3	10	5
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

We see that the differences between successive  $\triangle$  numbers are *not all the same*. Hence, this rule is *not linear*. It would be impossible to write it in the form

$$(\square \times a) + b = \triangle.$$

We could ask *this same question* in what looks like a different form, "Fill in the missing numbers so that the open sentence

$$(\square \times \square) + 1 = (\square \times \square) +$$

$\uparrow$                        $\uparrow$   
 Missing "definite, fixed" numbers

will become an *identity*." The answer is that this task is impossible. You cannot do it. (By contrast, if instead you had been asked to fill in the missing numbers so that the open sentence

$$(\square + 3) \times (\square + 3)$$

$$= (\square \times \square) + (\square \times \square) +$$

$\uparrow$                        $\uparrow$

would become an identity, you *could* do it; namely, as follows:

$$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9.$$

Evidently, then, our work on "guessing functions" *could* be related to our work on *identities* and *derivations*. However, we strongly recommend that you do not do this. It would be too "academic" for most children. There *are*, however, many lovely relations lurking just beneath the surface that some students may be lucky enough to discover. Let the discovery be its own reward — well, you might want to look honestly impressed if you *feel* honestly impressed by any such student discoveries.)

Let's return now to Ranny's rule in question 1. The table was

$\square$	$\triangle$	
0	3	Differences
1	5	2
2	7	2

If the differences *continue* to be 2 as we extend the table — then we shall know that *Ranny's rule is linear*, and we *can* write it in the form

$$(\square + a) + b = \triangle.$$

Indeed, by looking at the pairs

$\square$	$\triangle$
0	3

we see at once that  $b$  must be 3:

$$\begin{aligned} (\square \times a) + b &= \triangle \\ 0 \rightarrow \square, 3 \rightarrow \triangle \\ (0 \times a) + b &= 3 \\ 0 + b &= 3 \\ b &= 3 \end{aligned}$$

It is also easy to see that  $a$  must be 2 and that the rule must be

$$(\square \times 2) + 3 = \triangle.$$

(Have you figured out, yet, how we knew that  $a$  must be 2?)

Let's recapitulate: For the students, at this stage, we would let them merely "mess around"\* with numbers until they somehow found the correct rule. We believe this is good experience for them, and should *precede* any later "systematic" discussions. For the teacher, however, a more systematic approach is possible, and may be convenient. It consists of guessing *separately* the form of the answer (for example, is it  $(\square \times a) + b = \triangle$ ,  $(\square \times \square) + c = \triangle$ ,

\*The words "mess around" in this connection are borrowed from Professor David Hawkins, one of the wisest philosophers and educators of our generation. Some people find these words inelegant. There is, however, an intellectual equivalent of finger painting, and children need to "mess around" with ideas quite as much as they do with colors.

$d \times (\square \times \square) + e = \triangle$ , or  $f \times (\square \times \square) + (g \times \square) + h = \triangle$ , or whatever) and, separately, guessing what the actual numbers  $a, b, c, d, \dots$  must be. Thus, with Ranny's rule, we say that the form must be

$$(\square \times a) + b = \triangle.$$

Once we had decided upon this form, we saw that the actual numbers must be  $3 \rightarrow b, 2 \rightarrow a$ , so Ranny's rule had to be

$$(\square \times 2) + 3 = \triangle.$$

If this discussion sounds too fancy for your taste, ignore it, and just guess the rules as the children guess them. That may work better, anyhow.

(2) Can you write a formula for Ranny's rule, using  $\square$ 's and  $\triangle$ 's?

(2)  $(\square \times 2) + 3 = \triangle$

(3) Nancy says Ranny's rule is

$$(\square + 3) + 2 = \triangle.$$

Do you agree?

(3) **No. Let's try out Nancy's rule to see if it works:**

$$0 \rightarrow \square: \begin{aligned} (0 + 3) + 2 &= \square \\ 3 + 2 &= \triangle \\ 5 &\rightarrow \triangle \end{aligned}$$

So Nancy's rule would give us a table that starts like this:

$\square$	$\triangle$
0	5

Since this is *not* like Ranny's rule, Nancy's rule must be wrong.

(4) Kathy says Ranny's rule is: "Whatever number they say, double it, and add three." What do you think?

(4) **Kathy is correct.**

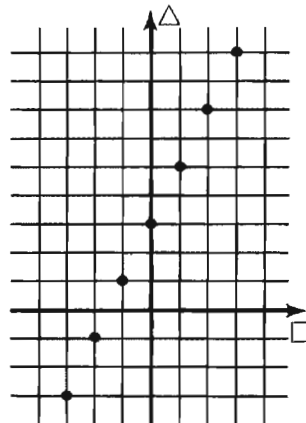
(5) Can you write a formula for Ranny's rule, using  $\square$ 's and  $\triangle$ 's?

(5)  $(\square \times 2) + 3 = \triangle.$

Notice that we now return to question 2, a standard bit of Madison Project "programming."

(6) Can you make a graph for Ranny's rule?

(6)



We know that a function (or rule) is *linear* if and only if it can be written in the form

$$(\square \times a) + b = \triangle,$$

where  $a$  and  $b$  are “definite, fixed” numbers that are agreed upon before the guessing starts. You might call this an *algebraic* criterion for linearity.

From a second point of view, we can look at the table, and say that a function is *linear* if and only if, when we arrange the  $\square$  numbers in sequence 0, 1, 2, 3, 4, . . . , the *differences* between successive  $\triangle$  numbers are all the same:

$\square$	$\triangle$	Differences	$\square$	$\triangle$	Differences
0	17	1	0	17	-3
1	18	1	1	14	-3
2	19	1	2	11	-3
3	20	1	3	8	-3
4	21	1	4	5	-3
5	22	.	5	2	.
.	.	.	.	.	.
.	.	.	.	.	.

You might call this an *arithmetic* criterion for linearity.

Now, the graph of question 6 gives us a third criterion for linearity: an equation is linear if and only if the graph of its truth set is a straight line.\* You might call this a *geometric* criterion for linearity.

If you reflect for a bit on what we have here, you can see why it is quite exciting and you will gain a deeper understanding of the “modern” mathematics curricula; for what we have is something that relates *arithmetic* to *algebra* and to *geometry*. The power of this kind of thing is lost when (as in traditional curricula) arithmetic, geometry, and algebra are studied separately. One important aspect of the “new” mathematics curricula is that we try to put mathematics together as a single, unified whole, rather than fragmenting it into little incomplete pieces by artificial lines of demarcation.

For emphasis, we can take two functions—one linear and one nonlinear—and look at them from all three points of view.

Linear Example

Nonlinear Example

An algebraic view:

$$(\square \times 3) + 2 = \triangle$$

$$(\square \times \square) - 1 = \triangle$$

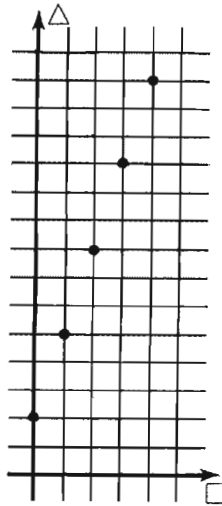
\*Purists will insist, “if and only if the graph of its truth set is a straight line which is not parallel to the  $\triangle$  (or vertical) axis.” I think you can safely ignore details of this kind most of the time, and nonetheless learn a great deal of valuable mathematics. Whenever these details become important—and usually their nuisance value greatly exceeds their honest importance—then that is the time to cope with them. They are not likely to be important in your work with children at this stage.



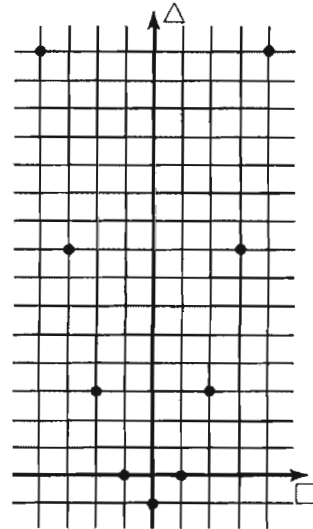
An arithmetical view:

□	△	Differences	□	△	Differences
0	2	5 - 2 = 3	0	-1	0 - (-1) = 1
1	5	8 - 5 = 3	1	0	3 - 0 = 3
2	8	11 - 8 = 3	2	3	8 - 3 = 5
3	11	14 - 11 = 3	3	8	15 - 8 = 7
4	14	⋮	4	15	⋮
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	Differences are all the same (i.e., always 3).	⋮	⋮	Differences are not all the same (1, 3, 5, 7, ...).

A geometrical view:



Graph points lie on a straight line.



Graph points do not lie on a straight line.

(7) Joan made up a rule. When Alan said "ten," Joan answered "fifty-three." When Alan said "five," Joan said "twenty-eight." Do you know Joan's rule?

(7) **Let's attack this problem by sheer guesswork, with no fancy theory. The pattern**

10 → 53  
5 → 28

suggests

10 → 50  
5 → 25,

and therefore suggests "multiplying by 5." Of course, this isn't quite right. In both cases we missed the target by 3: that is, we got 50 instead of 53 and 25 instead of 28. This suggests adding 3. So let's guess that Joan's rule is: Whatever number you tell me, I'll multiply it by 5, and then add 3. To settle the matter definitely, we would now need to ask Joan if this really is the rule she is using.

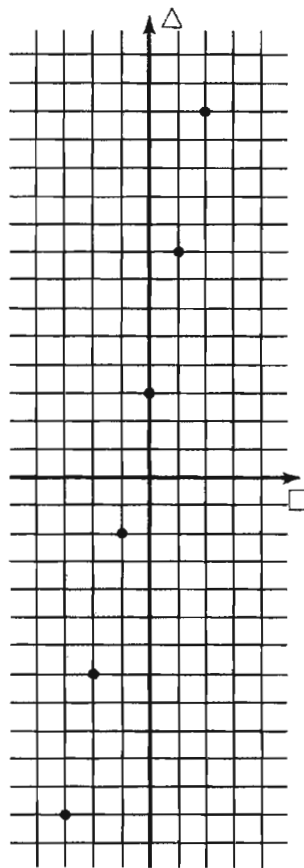
(8) Can you write a formula for Joan's rule?

(9) Can you make a graph for Joan's rule?

(8) The formula for this in  $\square, \triangle$  notation, would be

$$(\square \times 5) + 3 = \triangle.$$

(9)



(10) Why don't you make up your own rule, and see if people can guess it.

(11) Do you know what mathematicians mean by a function?

(12) Do you know the difference between a formula and a function?

(10) This will depend on your class.

You might wish to view the film entitled "Guessing Functions."

(11) See the discussion at the beginning of this chapter.

We strongly recommend that you underplay this, or even omit it entirely, rather than overdo it with your children. But if you can keep it simple, it is worth trying.

(12) You might say that

$$(\square \times \square) + \square = \triangle$$

and

$$\square \times (\square + 1) = \triangle$$

(or, better, use examples that arise naturally in your class) are two different formulas, but they represent the same function.

## GUESSING FUNCTIONS: FORM VS. NUMBERS

In the preceding chapter, some children made up "rules," and other children tried to guess these rules. (In our own classes, as you can see in the film "Guessing Functions," we let children guess first by describing the rule in words, and if they get this right, they then try to write the formula in  $\square$ ,  $\triangle$  notation. If they get the formula right, they select a couple of other students to help them make up the next rule, and the game continues. All of this is—we hope—made clear in the film.) The work of the preceding chapter—as far as the children are concerned—was a matter of informal "guessing." Children are invariably good at this, and enjoy it. We might call this "experience with functions" or "readiness-building for functions."

In the Teacher's Text, we presented a sketch of a somewhat more systematic approach to determining what rule the children were using. *We would not ordinarily tell this to our students!*

The essential feature of a systematic approach is to separate the *form* of a rule from the *actual numbers* used. In particular, we saw how to recognize the form

$$\begin{array}{c} (\square \times \text{---}) + \text{---} = \triangle \\ \uparrow \qquad \qquad \uparrow \\ \text{Some definite} \quad \text{Some definite} \\ \text{number} \qquad \qquad \text{number} \end{array}$$

and to distinguish it from other forms (for example, by looking at the differences between successive  $\triangle$  numbers in a table, or by marking points on a graph and seeing if they did, or did not, lie along a straight line). This is a form which children often use, but it is *not* the only form they use. (Incidentally, mathematicians call this kind of rule "linear.")

Here are some *different* forms children sometimes use, which are *not* linear:

$$\begin{array}{l} \square \times \square \times \square = \triangle \\ \square \times \square = \triangle \\ (\square \times \square) + 7 = \triangle \\ 36 - (\square \times \square) = \triangle \\ \frac{36}{\square} = \triangle \\ \vdots \\ \vdots \end{array}$$

The idea of trying to guess the *form* and the *actual numbers* separately is very powerful. In the present chapter, we shall try to suggest this approach to the children. (The methods shown in this chapter really were made up by a ninth-grade class at Nerinx High School in Webster Groves, Missouri.)

*This chapter is optional! Omit it if you wish!*



## CHAPTER 26

## Guessing Functions: Form vs. Numbers

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The students in a class at Nerinx High School, in Webster Groves, Missouri, have worked out some methods to help them guess functions. If you think these methods would help you, you may want to glance at the following few pages.

### I. IS THE FUNCTION LINEAR OR NOT?

Sometimes people make up a rule somewhat like these:

$$(\square \times 3) + 5 = \triangle$$

$$(\square \times 2) + -3 = \triangle$$

$$(\square \times 5) + -1 = \triangle$$

and so on.

In general, any rule of this sort looks like this:

$$(\square \times \text{---}) + \text{---} = \triangle$$

(Some  
number  
here)

(Some  
number  
here)

Rules (or functions) of this kind are called **linear functions**.

Do you know why?

(1) If the rule we are trying to guess is of this form, we can tell at once by making a graph and looking at it. What will the graph look like?

(2) How can we tell by looking at a table?

## ANSWERS AND COMMENTS

(1) The points will lie along a straight line. (Actually, along a line *not* parallel to the vertical, or  $\triangle$ , axis, but we don't usually mention this detail at this stage.)

(2) If we arrange the  $\square$  numbers in sequential order 0, 1, 2, 3, . . . ,

$\square$	$\triangle$
0	
1	
2	
3	
4	
⋮	
⋮	

and if we then look at the *differences* between successive  $\triangle$  numbers, they will all be the same if the rule is linear. Otherwise they will not be.

Example 1.

$\square$	$\triangle$	Differences
0	7	
1	11	$11 - 7 = 4$
2	15	$15 - 11 = 4$
3	19	$19 - 15 = 4$
4	23	$23 - 19 = 4$
5	27	$27 - 23 = 4$
⋮	⋮	⋮
⋮	⋮	⋮

The differences *are* all the same (in this case, 4), so the rule is linear and can be written in the form

$$(\square \times \text{Some definite number}) + \text{Some definite number} = \triangle.$$

Example 2.

$\square$	$\triangle$	Differences
0	3	
1	4	$4 - 3 = 1$
2	6	$6 - 4 = 2$
3	10	$10 - 6 = 4$
4	18	$18 - 10 = 8$
⋮	⋮	⋮
⋮	⋮	⋮

The differences are *not* all the same (being 1, 2, 4, 8, ...), and so the rule is *not* linear. It *cannot* be written in the form

$$(\square \times \text{Some definite number}) + \text{Some definite number} = \triangle.$$

We might summarize all of this by saying:

If the function *is* linear, the graph will be a straight line, the  $\triangle$  differences will all be the same, and it *can* be written in the form  $(\square \times a) + b = \triangle$ , where  $a$  and  $b$  are fixed, definite numbers. If the function is *not* linear, the graph will *not* be a straight line, the  $\triangle$  differences will *not* all be the same, and it *cannot* be written in the form  $(\square \times a) + b = \triangle$ , where  $a$  and  $b$  are definite fixed numbers.

Which of these functions are linear?

(3)

$\square$	$\triangle$
0	4
1	7
2	10
3	13
4	16
⋮	⋮
⋮	⋮

(3)

$\square$	$\triangle$	Differences
0	4	
1	7	$7 - 4 = 3$
2	10	$10 - 7 = 3$
3	13	$13 - 10 = 3$
4	16	$16 - 13 = 3$
⋮	⋮	⋮
⋮	⋮	⋮

The differences *are* all the same; the function *is* linear; its graph will be a straight line; it *can* be written in the form

$$\left(\square \times \frac{\quad}{\quad}\right) + \frac{\quad}{\quad} = \triangle.$$

↑ Some definite number     ↑ Some definite number

As a matter of fact, a formula representing this function is

$$(\square \times 3) + 4 = \triangle.$$

(You can verify this by checking this formula against the table shown above.)

(4)

□	△
0	3
1	5
2	8
3	12
⋮	⋮
⋮	⋮

(4)

□	△	Differences
0	3	
1	5	5 - 3 = 2
2	8	8 - 5 = 3
3	12	12 - 8 = 4
⋮	⋮	⋮
⋮	⋮	⋮

The differences are *not* all the same (2, 3, 4, ...); the function is *not* linear; its graph will *not* be a straight line; it *cannot* be written in the form

$$\left(\square \times \frac{\quad}{\quad}\right) + \frac{\quad}{\quad} = \triangle.$$

↑ Some definite number     ↑ Some definite number

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(5)

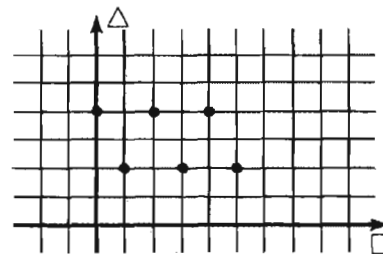
□	△
0	4
1	2
2	4
3	2
4	4
5	2
⋮	⋮
⋮	⋮

(5) The differences are *not* all the same (some are +2, others are -2); the function is *not* linear; its graph is *not* a straight line; it *cannot* be written in the form

$$\left(\square \times \frac{\quad}{\quad}\right) + \frac{\quad}{\quad} = \triangle.$$

↑ Some fixed number     ↑ Some fixed number

As a matter of fact, its graph is



A formula representing this function is

$$3 + (-1)^\square = \triangle.$$

You can verify this formula by working it out:

$$0 \rightarrow \square \quad 3 + (-1)^0 = 3 + 1 = 4$$

$$1 \rightarrow \square \quad 3 + (-1)^1 = 3 + -1 = 2$$

$$2 \rightarrow \square \quad 3 + (-1)^2 = 3 + +1 = 4$$

$$3 \rightarrow \square \quad 3 + (-1)^3 = 3 + -1 = 2$$

If you chose to omit Chapter 24, and have not studied *exponents* elsewhere, you may prefer to skip this example.

(6)

$\square$	$\triangle$
0	1
1	2
2	3
3	4
$\vdots$	$\vdots$
$\vdots$	$\vdots$

(6) The differences are all the same (namely, +1), so the function is linear. It can be written in the form

$$(\square \times \text{some number}) + \text{some number} = \triangle.$$

We now know the *form* of the answer; all that remains is to find the actual numbers.

(7)

$\square$	$\triangle$
0	17
1	14
2	11
3	8
4	5
5	2
6	-1
$\vdots$	$\vdots$
$\vdots$	$\vdots$

(7) The differences *are* all the same (namely, -3). Hence the function *can* be written in the form

$$(\square \times \text{some number}) + \text{some number} = \triangle.$$

Since we now know the *form* of the answer, all that remains is to find the actual numbers.

Can you find these two "missing" numbers by studying the following table?

$\square$	$\triangle$
0	17
1	14
2	11
3	8
$\vdots$	$\vdots$
$\vdots$	$\vdots$

(8)

$\square$	$\triangle$
0	1
1	2
2	5
3	10
4	17
5	26
10	101
100	10,001
1000	1,000,001
$\vdots$	$\vdots$
$\vdots$	$\vdots$

(8) The differences are *not* all the same (since the first was 1, and the second was 3), so the function is *not* linear. It *cannot* be written in the form

$$(\square \times \text{Some definite fixed number here}) + \text{Some definite fixed number here} = \triangle.$$

## II. IS THE FUNCTION EVEN OR ODD?

Some functions have tables like this:

$\square$	$\triangle$
1	4
-1	4
2	28
-2	28
3	108
-3	108
$\vdots$	$\vdots$
$\vdots$	$\vdots$

For these functions, 1 gets the same answer that -1 does, 2 gets the same answer that -2 does, and so on. Such functions are called **even functions**.

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Some functions have tables like this:

$\square$	$\triangle$
1	2
-1	-2
2	10
-2	-10
3	30
-3	-30
$\vdots$	$\vdots$
$\vdots$	$\vdots$

In this table, the answer for -1 is the **additive inverse** of the answer for 1, the answer for -2 is the additive inverse of the answer for 2, and so on. Functions like this are called **odd functions**.

Some functions are **neither even nor odd**. Here is one:

$\square$	$\triangle$
1	5
-1	-1
2	8
-2	-4
$\vdots$	$\vdots$
$\vdots$	$\vdots$

Which of these functions are even, which are odd, and which are neither even nor odd?

(9)

$\square$	$\triangle$
1	6
-1	6
2	9
-2	9
3	14
-3	14
$\vdots$	$\vdots$
$\vdots$	$\vdots$

(9) Even



(10)

$\square$	$\triangle$
1	2
-1	-4
2	5
-2	-7
3	8
-3	-10
$\vdots$	$\vdots$

(10) Neither even nor odd

(11)

$\square$	$\triangle$
1	1
-1	-1
2	14
-2	-14
3	51
-3	-51
$\vdots$	$\vdots$

(11) Odd

(12)  $y = x^2$ 

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(12) Even

(13)  $(\square \times \square) = \triangle$ 

(13) Even

(14)  $(\square \times \square) \times \square = \triangle$ 

(14) Odd

(15)  $y = x^3$ 

(15) Odd

(16)  $y = 5x^2$ 

(16) Even

(17)  $(3 \times \square) + 2 = \triangle$ 

(17) Neither even nor odd

(18)  $y = 3x + 2$ 

(18) Neither even nor odd

(19)  $y = x^4$ 

(19) Even

(20)  $y = x^4 - x^2$ 

(20) Even

(21)  $y = 3x^2 + 2$ 

(21) Even

(22)  $y = x^3 + x$ 

(22) Odd

(23) How can the idea of even and odd functions help you to guess functions?

(23) If the table indicates an even function, then the formula may be a sum of even powers, as in

$$x^4 + x^2 = y \quad (\text{or } \square^4 + \square^2 = \triangle)$$

$$x^6 - 3x^2 + 8 = y \quad (\text{or } \square^6 - 3 \times \square^2 + 8 = \triangle)$$

$$\vdots$$

If the table indicates an *odd* function, then the formula may be a sum of odd powers:

$$\begin{array}{ll} x^3 = y & (\square^3 = \triangle) \\ x^3 - x = y & (\square^3 - \square = \triangle) \\ 7x^3 - 2x = y & (7 \times \square^3 - 2 \times \square = \triangle) \\ x^5 + 8x^3 + 2x = y & (\square^5 + 8 \times \square^3 + 2 \times \square = \triangle) \\ \vdots & \vdots \end{array}$$

If the table indicates that the function is *neither even nor odd*, then the formula may be of a sum of some even powers plus some odd powers:

$$\begin{array}{ll} x^2 + x = y & (\square^2 \times \square + \square = \triangle) \\ x^4 - x^3 = y & (\square^4 - \square^3 = \triangle) \\ x + 2 = y & (\square + 2 = \triangle) \\ 7x + 8 = y & (7 \times \square + 8 = \triangle) \\ 7x + 3 = y & (7 \times \square + 3 = \triangle) \\ \vdots & \vdots \end{array}$$

Notice that: 2 is an *even* power ( $2 = 2 \cdot x^0$ , so the *exponent* is 0, which is even); 3 is (in *this* sense) an even function ( $3 = 3 \cdot x^0$ , so that again the *exponent*, which is the decisive point, is even)\*;  $x$  is an *odd* function ( $x = x^1$ , so the *exponent*, which is 1, is odd);  $4x$  is (in this sense) an *odd* function ( $4x = 4 \cdot x^1$ , so again the all-decisive *exponent* is 1, and is therefore odd);  $3x^2$  is an *even* function (since the *exponent* 2 is even);  $4x^2$  is an even function (since the *exponent* 2 is even); and so on.

### III. ELLEN'S METHOD

Sometimes, when she was trying to guess a rule, Ellen would say,

"Use your rule on 10, and tell me the answer."

When she got the answer to this, she would say,

"Use your rule on 100, and tell me the answer."

When she got this answer, she would request,

"Use your rule on 1000, and tell me the answer."

(24) Why do you suppose Ellen picked these numbers to ask about?

(24) Actually, using  $\square$  numbers such as 10, then 100, then 1000, and so on, helps to do two things: it reveals the "rate of growth" of the function—an important mathematical idea which we shall explain presently—and it tends to separate the "dominant" terms from the "minor" terms—again, ideas which we shall explain in a moment.

\*We have called  $y = 3$  an *even function*, in the sense of functions. Of course, 3—regarded as a positive integer in the sense of number theory—would be an *odd integer*. Different branches of mathematics often use words differently (unfortunately!), so it often becomes important to know which branch of mathematics is involved. In the present instance, when we are thinking in terms of function theory and are classifying functions as "even," "odd," or "neither," then 3 is an even function. When we are thinking in terms of number theory and are classifying integers as "even" or "odd," then 3 is an odd integer.

The easiest way to see what mathematicians mean by *dominant terms* is to look at a simple example: Consider the function  $2x + 3 = y$ . Here is part of its table:

x	y
1	5
10	23
100	203
1000	2003
10,000	20,003
⋮	⋮
⋮	⋮

From looking at this table, you can probably guess what mathematicians mean by "dominant terms." When we used UV ( $UV: 1000 \rightarrow x$ ), we got  $y = 2003$ . Here,  $y$  is approximately twice  $x$ ; after all, 2003 is very nearly equal to 2000. (A discrepancy of 3 in a number as large as 2000 is almost never important in ordinary engineering or measurement, unless—for some special reason—unusual precision is necessary. If two cities are 2003 miles apart, they are, for almost all sensible purposes, 2000 miles apart, and so on.)

For  $x = 1000$ , the term  $2x$  *dominates* the term 3. If we use an even larger value of  $x$  ( $UV: 10,000 \rightarrow x$ ), the "dominance" becomes even clearer:  $y = 20,003$ . A discrepancy of 3 in a number as large as 20,000 is (for nearly all ordinary purposes) even less significant than an error of 3 in a number as large as 2000. The "dominance" of the term  $2x$  over the term 3 is even more pronounced.

If we used still larger values for  $x$ , the dominance of the  $2x$  term would become still clearer. (You may want to consider this question: If we used very small positive values for  $x$ —such as  $UV: \frac{1}{100} \rightarrow x$  or  $UV: \frac{1}{1000} \rightarrow x$  or  $UV: \frac{1}{10000} \rightarrow x$ —which term would be "dominant," the term  $2x$  or the term 3?)

We shall not try to put into words explicitly what we mean by dominant terms. Here, quickly, are a few more examples:

$$x^3 - x^2 + 7 = y \quad \text{If } x \text{ is large, the dominant term is } x^3.$$

$$x + 5 + \frac{1}{x} = y \quad \text{If } x \text{ is large, the dominant term is } x.$$

(You might say, roughly, that you would happily settle for the  $x^2$ , and forget about the  $-x^2 + 7$ , if  $x$  is a large number. This somehow suggests Mark Twain's remark that he would gladly do without the necessities of life, provided he could have all of the luxuries.)

Once we are able to "separate out" the *dominant term* in the "rule" for large values of  $x$ , then we can easily see what is meant by *rate of growth*.

For the linear function  $y = 2x + 3$ , the table, as we just saw, was

x	y
1	5
10	23
100	203
1000	2003
10,000	20,003
100,000	200,003
⋮	⋮
⋮	⋮

Evidently, if  $x$  is a large number, it is approximately true that whenever we multiply  $x$  by 10, the new  $y$ -value becomes 10 times as large as the previous one.

For example:

$x$	$y$
10,000	20,003
100,000	200,003

Take a value of  $x$  10 times larger

and you get back a new  $y$ -value that is (approximately) 10 times larger.

In such a case, we say that (approximately)  $y$  grows at the same rate that  $x$  does.

For the function

$$y = x^2 + 7,$$

here is part of the table:

$x$	$y$
1	8
10	107
100	10,007
1000	1,000,007
⋮	⋮
⋮	⋮

For the function (or "rule"), we see that whenever we multiply  $x$  by 10, the corresponding new  $y$ -value is 100 times as large as the previous  $y$ -value.

For example:

$x$	$y$
100	10,007
1000	1,000,007

Multiply  $x$  by 10.

Now, 1,000,000 is  $100 \times 10,000$  (using simpler, approximate numbers).

When the table shows this pattern, it is usually a clue that the *dominant term involves  $x^2$* , and not  $x$  (or  $x^3$ , etc.). Hence, if we see this pattern in the table, we can reasonably guess that the rule has a form like

$$\frac{\text{Some definite number}}{\uparrow} x^2 + \frac{\text{Some definite number}}{\uparrow} x + \frac{\text{Some definite number}}{\uparrow} = y$$

or perhaps like this

$$\frac{\text{Some definite number}}{\uparrow} x^2 + \frac{\text{Some definite number}}{\downarrow} \frac{1}{x} = y$$

We would start looking for rules like

$$2x^2 - 3x + 5 = y$$

$$x^2 + x + 7 = y$$

$$x^2 + \frac{3}{x} = y$$

$$10x^2 - \frac{1}{x} = y$$

⋮

Let us now try "Ellen's method" on the rule in problem 25.

(25) Using her method, Ellen got this table:

□	△
1	2
10	100.1
100	10,000.01
1000	1,000,000.001
⋮	⋮

Can you guess this function?

(25) We can notice several things:

(i) Evidently, the △ numbers consist of two parts, the dominant term and a small "correction" term. Let's rewrite the table, to show this even more clearly:

□	△	Dominant term in △ number	Small "correction" term in △ number
1	2	impossible to tell	
10	100.1	probably 100	probably 0.1
100	10,000.01	10,000	0.01
1000	1,000,000.001	1,000,000	0.001
⋮	⋮	⋮	⋮

As we use larger □ numbers, the pattern becomes clearer.

(ii) Looking only at the dominant term, we have:

□	Dominant term in △ number
10	100
100	10,000
1000	1,000,000

Here we see the *rate of growth* rather clearly. When we multiply the □ number by 10, the corresponding △ number (or, at least, its dominant term) becomes multiplied by 100. This suggests that the dominant term involves

$$\square \times \square.$$

Hence, we shall look for forms such as

$$\begin{array}{c} \text{---} \\ \uparrow \\ \text{Some} \\ \text{definite} \\ \text{number} \end{array} \times (\square \times \square) + \begin{array}{c} \text{---} \\ \uparrow \\ \text{Some} \\ \text{definite} \\ \text{number} \end{array} \times \square + \begin{array}{c} \text{---} \\ \uparrow \\ \text{Some} \\ \text{definite} \\ \text{number} \end{array} = \triangle$$

Or perhaps we shall look for forms such as

$$\begin{array}{c} \text{Some definite number} \\ \uparrow \\ \square \end{array} \times (\square \times \square) + \frac{\begin{array}{c} \text{Some definite number} \\ \text{as numerator of this fraction} \\ \downarrow \\ \square \end{array}}{\square} = \triangle$$

To give some more specific examples, we shall be looking for rules of the same general type as

$$\begin{aligned} [2 \times (\square \times \square)] + 7 &= \triangle \\ (\square \times \square) + (3 \times \square) &= \triangle \\ (\square \times \square) + \frac{2}{\square} &= \triangle \\ &\vdots \end{aligned}$$

(iii) But we can see even more from our table! Let's look at the "small correction term" which marks the difference between the actual  $\triangle$  numbers and the dominant term in the  $\triangle$  numbers:

$\square$	Small "correction" term in $\triangle$ number (representing "minor" terms)
10	0.1
100	0.01
1000	0.001
$\vdots$	$\vdots$
$\vdots$	$\vdots$

This is reasonably obvious. The minor term seems to be

$$\frac{1}{\square}$$

Hence, we guess

$$(\square \times \square) + \frac{1}{\square} = \triangle$$

(This is, in fact, the correct formula for the "rule" in problem 25.)

As written out in detail, this all seems rather cumbersome. Relax—trust your "hunches" or your "intuition"—and I believe you will find that "guessing functions" is both easy and enjoyable. There is a system—or at least fragments of a system—but don't try to be too systematic: it spoils the fun.

(26) This eliminates the forms

$$y = ax + \frac{b}{x}$$

and

$$y = ax^2 + \frac{b}{x}$$

#### IV. LOOKING FOR THE RIGHT FORM

Another very good method, which the Nerinx students often use, goes like this:

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First try to list several likely forms. For example, these forms are often used:

$$\begin{aligned} y &= ax + b \\ y &= ax + \frac{b}{x} \\ y &= a + bx + cx^2 \\ y &= ax^2 + \frac{b}{x} \end{aligned}$$

In these forms,  $a$ ,  $b$ , and  $c$  are definite numbers, chosen by the people who made up the rule. The letter  $x$  indicates "the number we tell them," and the letter  $y$  represents "the answering number that they tell us."

Once you have written your list of likely forms, try to ask questions which will eliminate some of them or confirm one of them. Here are some questions the Nerinx students often use:

(26) "Use your rule on 0, and tell me the answer." Suppose the answer is 3.

$x$	$y$
0	3

Which forms on the list preceding would be eliminated by this answer?

since division by zero would have been required with these forms. If the students had been using either of these forms, they should have answered,

"Zero doesn't work in our rule."

(27) "Use your rule on 0, and tell me the answer."  
Suppose the answer is: "Our rule doesn't work for 0."

$x$	$y$
0	no answer

Which forms on the list would be eliminated by this answer?

(28) Suppose the table was:

$x$	$y$
1	9
-1	5
2	11
-2	3
⋮	⋮
⋮	⋮

What forms does this table suggest?

(27) See answer to question 26. This time the rules  $ax + b = y$  and  $a + bx + cx^2 = y$  are eliminated.

(28) Let's look first for successive differences. In order to do this, we must arrange the  $x$  numbers in proper sequential order:

	$x$	$y$	
	-2	3	
x increases by 1.	-1	5	y increases by 2.
	1	9	
x increases by 1.	2	11	y increases by 2.

These are too few values to settle the matter definitely, but it looks as if the rule is *linear*:

$$y = ax + b,$$

where  $a$  and  $b$  are some definite numbers. Moreover, when we increase  $x$  by 1,  $y$  appears to increase by 2, so that it seems likely that

$$a = 2.$$

Moreover, if the linear pattern holds, then we would expect that, if we told them 0, they would answer with a number half-way between 5 and 9.

$x$	$y$
-2	3
-1	5
0	←
1	9
2	11

That is to say, we expect they would answer 7. Hence, the formula seems to be

$$2x + 7 = y.$$

Notice that this was by no means the only way to think about this problem. You might, instead, have noticed:

(i) The function is neither even nor odd; therefore it is probably the sum of an even term plus an odd term.

(29) Suppose the table was:

$x$	$y$
1	-2
-1	-2
2	1
-2	1

What forms does this table suggest?

As you begin to see what form the function probably has, you can try to find the actual numbers—that is, the numerical replacements for the variables  $a$ ,  $b$ ,  $c$ , etc., in the form.

(ii) You could also have guessed at the rate of growth, although this is a bit difficult when we have only used small numbers.

Still other methods would have been possible. For example, you could have made a graph, etc.

(29) Perhaps the first thing we notice about this table is that it seems to represent an even function:

- UV:  $1 \rightarrow x$  yields  $y = -2$ .
- UV:  $-1 \rightarrow x$  yields  $y = -2$ , which is the same as for  $x = 1$ .
- UV:  $2 \rightarrow x$  yields  $y = 1$ .
- UV:  $-2 \rightarrow x$  yields  $y = 1$ , again, the same as for  $x = 2$ .

Hence, we might guess the form

$$\begin{array}{c}
 \text{---} x^2 + \text{---} = y. \\
 \uparrow \qquad \qquad \uparrow \\
 \text{Some} \qquad \qquad \text{Some} \\
 \text{definite} \qquad \qquad \text{definite} \\
 \text{number} \qquad \qquad \text{number}
 \end{array}$$

To go on from here, we might say, "Use your rule on zero, and tell us the answer."

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(30) Can you find this function?

$x$	$y$
0	0
1	0
-1	0
2	30
-2	-30
3	240
-3	-240
10	99,990
100	9,999,999,900
⋮	⋮
⋮	⋮

(30) We can see immediately, from the table, that we seem to be dealing with an odd function. This suggests forms of the same general type as

- $7x = y$
- $x^3 = y$
- $2x^3 = y$
- $2x^3 + 7x = y$
- $x^5 = y$
- $x^5 - x = y$
- ⋮
- ⋮

Let's try to find the dominant term:

$x$	$y$	Dominant term in $y$	Small "correction" term, or "minor" term
0	0	hard to tell	?
1	0		
-1	0		
2	30		
10	99,990	apparently 100,000	apparently -10
100	9,999,999,900	apparently 10,000,000,000	apparently -100
⋮	⋮		
⋮	⋮		



It appears immediately that the *minor* term is  $-x$ , so we have a form *generally* of this type:

$$\begin{array}{c} \text{---} x^3 - x = y \\ \uparrow \\ \text{Some definite number} \\ \text{---} x^5 - x = y \\ \uparrow \\ \text{Some definite number} \\ \text{---} x^7 - x = y \\ \uparrow \\ \text{Some definite number} \\ \vdots \\ \vdots \end{array}$$

Can we tell whether this first term is  $x^3$ ,  $x^5$ , or  $x^7$ , or whatever? Yes, we can, by looking at the *rate of growth* of the function.

$x$	$y$	Apparent dominant term in $y$
10	99,990	100,000
100	9,999,999,900	10,000,000,000

When you multiply  $x$  by 10

the dominant  $y$ -term becomes multiplied by 100,000.

Since  $100,000 = 10^5$ , this suggests that the dominant term involves  $x^5$ . Putting all of this together, we might guess that the "rule" is

$$x^5 - x = y,$$

which is, in fact, correct.

## WHERE DO FUNCTIONS COME FROM?

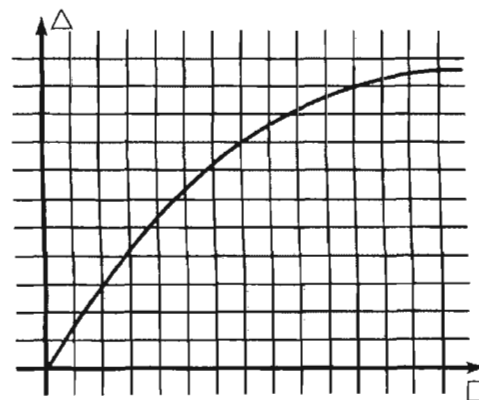
In the two preceding chapters, some students made up a "rule" – actually a *mathematical function* – and we tried to guess their rule. Functions, however, need not come from a conspiracy of our colleagues, as they did in Chapter 25. They may also come from a wide variety of physical situations, social situations, and so on. In this chapter we shall study some functions obtained from various puzzles and games, from hanging weights on springs, and so on.

In general, our task is to approach some situation or apparatus, and to try to achieve some "understanding" of it. Specifically, we shall try to make up some "model," or "cognitive structure," that will crudely represent the complex reality in terms simple enough for us to be able to think about. This will call for three mathematical tools:

- (i) The *table*, as in

$\square$	$\triangle$
7	3
9	2
11	1
12	$\frac{1}{2}$

- (ii) The *graph*, as in



- (iii) The *formula (or equation)*, as in

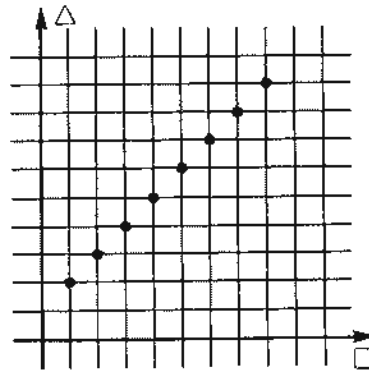
$$(\square \times 3) + 7 = \triangle.$$

Because these will be our three principal theoretical tools, it is useful to ask how, if we are given one, we may obtain the others. In some cases this is very easy, while in other cases it can be quite hard.

Suppose, for example, we had obtained this table:

$\square$	$\triangle$
1	2
2	3
3	4
4	5
$\vdots$	$\vdots$
$\vdots$	$\vdots$

If we now wish to start from this table, and represent this same function by a graph, our task is easy enough. We have only to plot the points (1, 2), (2, 3), and so on, on Cartesian coordinates, quite as if we were playing tic-tac-toe. Here we have this same function, represented by a graph:



It is clear that, in general, starting from a table and making a graph is quite easy.

End up with the same function represented by:

		Table	Graph	Equation
Start with the function represented by:	Table		usually easy	
	Graph			
	Equation			

Similarly, if we start with an equation, we can easily construct a table, by merely making numerical replacements for the variables (the process we call UV):

$$\begin{aligned}
 & (\square \times 1) + 3 = \triangle \\
 \text{UV: } 1 \rightarrow \square & \quad (1 \times 1) + 3 = \triangle \\
 & \quad \quad \quad 1 + 3 = \triangle
 \end{aligned}$$

Evidently, to obtain a *true* statement we must put 4 into the  $\triangle$ :

$$4 \rightarrow \triangle.$$

Hence we have

□		△
1		4

Proceeding in the same way, with  $2 \rightarrow \square$ , then with  $3 \rightarrow \square$ , and so on, we can construct the table:

□		△
1		4
2		5
3		6
4		7
5		8
⋮		⋮
⋮		⋮

(As usual, *discovering patterns* can make our work easier.)

If we think this example is typical, then we can fill in another square in our chart:

		End with:		
		Table	Graph	Equation
Start with:	Table	/	usually easy	
	Graph		/	
	Equation	usually easy		/

You and your class may wish to explore further the problem of filling in other squares on this chart. How, for example, would you *start with a table* and obtain from it *an equation*?

This topic of "crossing back and forth" between tables, graphs, and formulas is one we have met, at least briefly, in Chapter 25 and elsewhere (see also *Discovery*, Chapters 11, 15, 17, 18, 35, and 49).

The three tools of *tables*, *graphs*, and *equations* are not the whole story, however. If we start with, say, weights hanging on springs, we might make a table by recording, successively, various *weights* and the *corresponding distance that the spring was stretched*. However, in measuring the distances that the spring stretched, we shall inevitably make errors. Rulers slip, we misread them, and sometimes we do not look "levelly" from the ruler to the spring, etc. (To make matters worse, the spring may jiggle a bit.)

Consequently, we may, if we wish, make use of what we learned about "measurement uncertainties" in Chapter 17. (You may also wish to view the film "Weights and Springs.")

There is a philosophical problem here that is worthy of a moment's thought. The *model* which we shall make for, say, the stretching spring will be oversimplified. Careful observation of real springs will show that their behavior is extremely complicated. (Actually, it depends upon many things, such as temperature, and even *depends upon the past history of that individual spring*. If we "mistreat" a spring, it will behave differently from then on.) Since the reality is complicated, and our model is relatively simple (just how simple is for us to decide), our model *only roughly* corresponds to the reality. In this sense, we might say that our model is "wrong" — and anybody else's model will be "wrong," too, although more complicated models may match the reality more closely. (This does not necessarily mean that they are better models, for more complicated models will have the *disadvantage* that they are harder to think about.)

Then, to make matters worse, our measurements of distances, times, weights, volumes, etc., will contain some inevitable measurement errors. In this sense, *our numbers will also be "wrong."*

Does this make matters hopeless? By no means! Our models are wrong, but *useful*. Our measurement numbers are wrong, but *useful*. (In a somewhat similar sense, one might say our schools are imperfect, but valuable. Whatever is worth doing is worth doing moderately well, if that's the best that is possible.)

It is worth recalling Piaget's ideas about *cognition*. All of our "knowledge" represents an oversimplification of reality, and in this sense *all of our knowledge is "wrong"* — at the very least, it is incomplete.

Now, by using expensive and intricate measuring devices, we can obtain numbers where the measurement errors are *smaller* — but still not zero! By developing extremely complicated models and mental imagery, we can match our mental imagery more closely to the reality — but there will still be differences! By taking great pains in our learning, thinking, and "understanding," we can bring our "knowledge" to an impressive level of sophistication. We can cope with monumental tasks, like photographing the remote side of the moon, or flying from St. Louis to New York in an hour and a half. *Nonetheless, even our best thought is "wrong."* It is imperfect, and, by suitably heroic efforts, it can be made better.

In this sense our plight is one of perpetual open-endedness. We never arrive at the final answer. However far we journey in our measuring, our thinking, our reflecting, our philosophizing, our studies . . . the unknown always lies ahead.\* The unknown . . . and the uncertain.

If newness, change, obsolescence, and a lack of ultimate answers do in fact characterize our age as much as they seem to, then this gives our schools a major task in educating our young people so that they can cope with a future which always extends beyond the horizon of our present vision.

This is, in large part, a *new* demand upon our schools — and upon those of us who educate the young. The eminent physicist Robert

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\*That this is true: even within the "pure" fields of *mathematics* and *logic* is indicated by some remarkable work of the great logician Kurt Gödel, of Princeton's Institute for Advanced Study.

Karplus, of the University of California at Berkeley, has introduced the word *lysophobia*, the "fear of leaving loose ends," to describe an attitude often found in our schools in the past. Lysophobia means *demanding* final answers to questions that we give to our students and that our students give to us. Of course, these are not really "final" answers—they are counterfeit "final" answers. For example: how many chemical elements are there? The "final" answer used to be 92. How many were there in June, 1965? "Whatever goes up comes down" used to be a very ultimate truth, if you will pardon the phrase, yet nowadays it may go into orbit about the sun, among other possibilities.

For the opposite of lysophobia, Karplus has introduced the word *lysophilia*, the "love of leaving loose ends." He argues, and many people are inclined to agree, that our schools must try to wean themselves from a traditional addiction to lysophobia, and to acquire instead a taste for lysophilia. It's probably somewhat like giving up smoking.

For two very valuable discussions of this matter, on a philosophical level, you may wish to pore thoughtfully over Polyani (134) and Teilhard de Chardin (33).

This idea has been growing gradually, and is not entirely new. Consider, for example, this excerpt from Ralph Waldo Emerson.\*

Where do we find ourselves? In a series of which we do not know the extremes, and believe that it has none. We wake and find ourselves on a stair; there are stairs below us, which we seem to have ascended; there are stairs above us, many a one, which go upward and out of sight.

I take this evanescence and lubricity of all objects, which lets them slip through our fingers then when we clutch hardest, to be the most unhandsome part of our condition. Nature does not like to be observed, and likes that we should be her fools and playmates. We may have the sphere for our cricketball, but not a berry for our philosophy. Direct strokes she never gave us power to make; all our blows glance, all our hits are accidents . . .

Dream delivers us to dream, and there is no end to illusion.

If you like the process of trying to study reality, you and your students may get great enjoyment from *Mathematics and Living Things*, School Mathematics Study Group (Stanford University, Stanford, California).

A very pleasant relation between the mathematical concept of *slope of a linear graph* and the physical concept of *density*—which moreover gives children experience with ratio and proportion—has been made into a teaching unit by Frederick L. Ferris, Jr., and his colleagues, of the Junior High School Science Project of Princeton University. For information, write to Professor Ferris.

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\*Ralph Waldo Emerson, *Essays and Poems*, selected and arranged by G. F. Maine, with an Introduction by DeLancey Ferguson (Collins, London and Glasgow, 1962)



CHAPTER 27

Where Do Functions  
Come From?

[page 84]

THE EASY THREE-PEG GAME

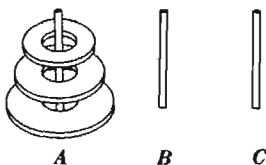
(1) Suppose you have three pegs:



and three washers, of three different sizes:

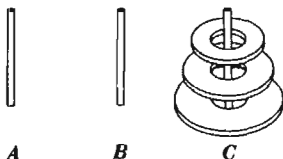


To play the game, you start with all three washers on peg A, with the largest washer on the bottom and the smallest washer on the top.



You move one washer at a time, taking it off one peg and putting it on another.

You are finished when you have all three washers on peg C, with the largest washer on the bottom and the smallest washer on the top.



Can you get the washers onto peg C this way?

[page 85]

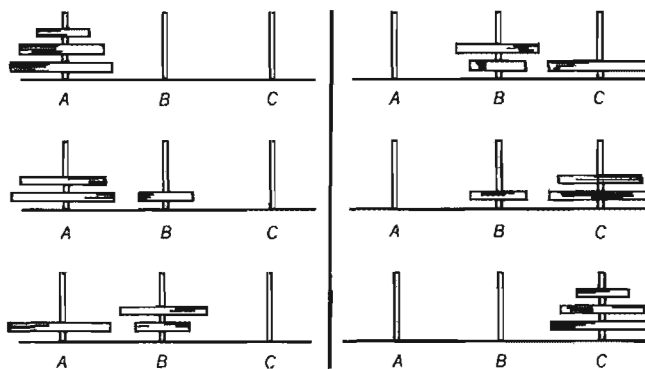
(2) How many moves did you need to get all three washers onto peg C?

(3) Could you have done it in fewer moves?

ANSWERS AND COMMENTS

We start with a "counting" function, rather than a "measuring" one, so as to free ourselves, temporarily, from a need to consider measurement errors.

(1) If you used the fewest possible moves, you proceeded like this:



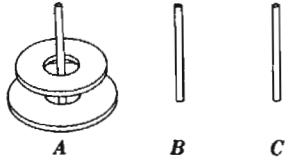
Since people do not usually count the initial starting position as a "move," you would probably count this as 5 moves.

(This is a simplified version of the ancient puzzle known as "the tower of Hanoi." The use of "the tower of Hanoi" in this context was suggested by Donald Cohen, of the Clayton, Missouri, public schools; this simplified version was suggested by Knowles Dougherty, of Webster College. The simplified version, as we shall see, leads to a straight-line graph—that is, to a "linear function." The original version, as we shall also see, does not give us a straight-line graph—mathematicians would say it gives us a "nonlinear function.")

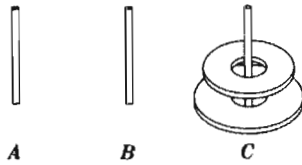
(2) If you used the fewest possible moves, you used 5.

(3) Not if you did it the way we did in question 1.

(4) Suppose we change the rules of the game. Everything else is the same, but we start with only two washers:



Can you get both washers onto peg C, with the little one on top?



(5) How many moves did you need?

(6) Jean made this table:

Number of washers	Minimum number of moves
2	?
3	?

Can you fill in the two missing numbers in Jean's table?

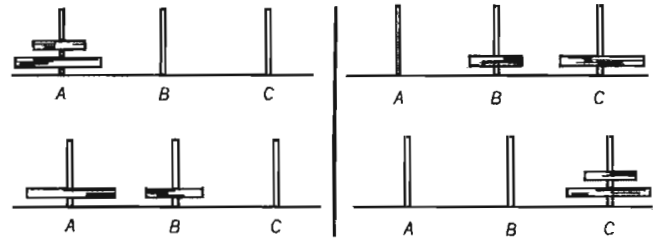
(7) What would happen if you used four washers?

(8) Can you extend Jean's table?

(9) What would happen if you used 100 washers?

(10) Can you write a formula for this function?

(4) Here is the whole story, starting with two washers:



(5) If you used as few as possible, you used 3.

(6) Number of washers

Minimum number of moves

Number of washers	Minimum number of moves
2	3
3	5

(7) You would require 7 moves:

Number of washers

Minimum number of moves

Number of washers	Minimum number of moves
2	3
3	5
4	7

(8) Here is an extension—you could, of course go further:

Number of washers	Minimum number of moves
1	1
2	3
3	5
4	7
5	9
6	11
7	13
⋮	⋮
⋮	⋮

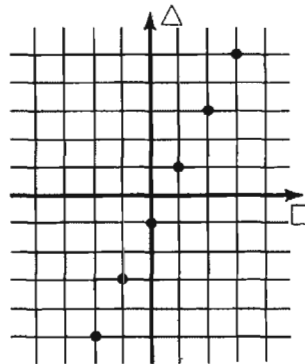
(9) You would require 199 moves.

(10)  $(\square \times 2) - 1 = \triangle$



(11) Can you make a graph for this function?

(11)



From looking at this graph\*, you can see why mathematicians call this a *linear* function.

(12) Does it work the same way if you start with only one washer?

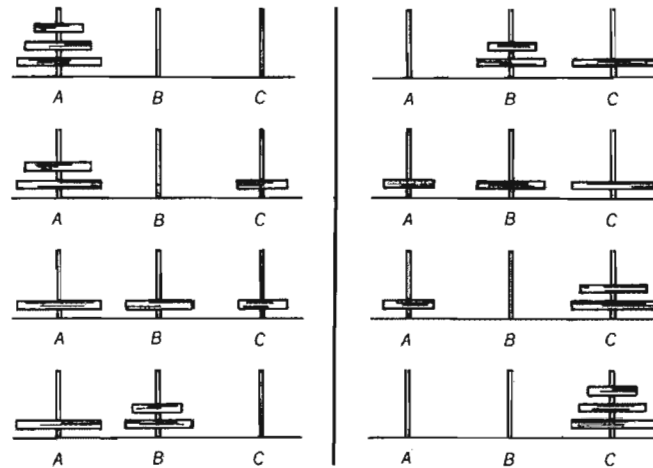
(12) Yes; the same general pattern holds. You require 1 move.

**THE HARD THREE-PEG GAME**

(13) You play this game with the same rules as the "easy three-peg game," but you add one additional rule: you must never, at any stage, put a larger washer on top of a smaller washer.

(13) We turn now to the original version, by adding the rule that you must never, in any move, place a larger washer over a smaller one. If you start with 1 washer or with 2 washers, you proceed exactly as in the "easy version." If you start with 3 washers, you can proceed like this:

With this additional rule, can you move the washers from peg A to peg C?



This, evidently, required 7 moves.

A table, summarizing our work thusfar, would look like this:

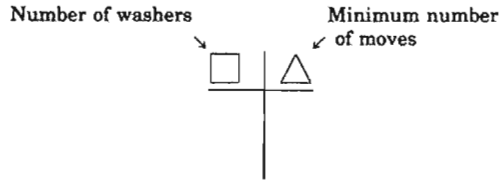
Number of washers	Minimum number of moves
1	1
2	3
3	7

The hard three-peg game

\*Note: Mathematicians have used the idea of "function" in several different ways. In the present three-peg game, the function matches the reality only when the number of disks is a positive integer. However, the notion of "extending systems" which we studied in Chapter 24 can be applied in many places, including here. In the graph shown, we have extended the function to include zero and negative values. Of course, we should not be surprised if this extended function fails to match the physical reality (as represented by the game).

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(14) Can you make a table for the "hard three-peg game"?



(15) Suppose you used 10 washers, how many moves would you need?

Before going further, you may want to pause and work out for yourself the case where you start with 4 washers, and the case where you start with 5 washers. By then, if you reflect on what you have done, you may be able to make up a complete "theory" for this game.

(14) Here is more of the table:

Number of washers	Minimum number of moves
1	1
2	3
3	7
4	15
5	31
6	63
⋮	⋮
⋮	⋮

The hard three-peg game

(15) One way to answer this would be to extend our table:

Number of washers	Minimum number of moves
1	1
2	3
3	7
4	15
5	31
6	63
7	?

Number of washers	Minimum number of moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023
⋮	⋮
⋮	⋮

So, the answer is, for 10 washers, we would require 1023 moves.

Notice the power of mathematics; it would have taken some time to work this out directly, even if you can count to 1023 without making any mistakes! Professor Warwick Sawyer has pointed out that one of the reasons for knowing arithmetic is so we can recognize *patterns* when they are lurking right before our noses. Well, there is another pattern here that we have not yet exploited. We will need it for the next problem. Can you find it?

(16) Suppose you used 100 washers, how many moves would you need?

(16) Our powerful method of problem 15 lets us *imagine* making 1023 moves without having to do it. Even that method, however, is not powerful enough to handle this problem easily. We need a more

powerful method still. Can we find one? Well, the place to look is usually for some *pattern* that can be put to use. Look at the numbers

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, \dots$$

Do they remind you of anything? They do, if you know enough *arithmetic*. One fundamental pattern of arithmetic is

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \dots$$

These are the "successive powers of 2":

$$\begin{array}{ll} 2 = 2 & 2^6 = 64 \\ 2^2 = 2 \times 2 = 4 & 2^7 = 128 \\ 2^3 = 2 \times 2 \times 2 = 8 & 2^8 = 256 \\ 2^4 = 2 \times 2 \times 2 \times 2 = 16 & 2^9 = 512 \\ 2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32 & 2^{10} = 1024 \end{array}$$

In the statement  $2^{10} = 1024$ , the small, raised numeral 10 is known as the *exponent*. The number  $2^{10}$  is read "2 to the tenth power," "2 with the exponent 10," or simply "2 to the tenth."

Notice that we can group factors like this:

$$(2 \times 2) \times (2 \times 2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2$$

Thus  $2^2 \times 2^3 = 2^5$ . In general, for whole numbers  $n$  and  $m$ , we have  $2^n \times 2^m = 2^{n+m}$ .

We can also group factors like this:

$$(2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

Thus  $2^2 \times 2^2 \times 2^2 = 2^6$ , or  $(2^2)^3 = 2^6$ .

Let's try out all these patterns on our problem, and see if they are powerful enough to let us handle it reasonably easily.

Evidently, our table is:

$\square$	$\triangle$
1	2 - 1
2	4 - 1
3	8 - 1
4	16 - 1
5	32 - 1
⋮	⋮
⋮	⋮

The table can also be written as:

$\square$	$\triangle$
1	$2^1 - 1$
2	$2^2 - 1$
3	$2^3 - 1$
4	$2^4 - 1$
5	$2^5 - 1$
⋮	⋮
⋮	⋮

Hence, further on in the table, we should find:

□	△
99	$2^{99} - 1$
100	$2^{100} - 1$
101	$2^{101} - 1$
⋮	⋮

This is exactly what we want! If we use 100 washers, we shall require  $2^{100} - 1$  moves! Now,  $2^{100} - 1$  is really our answer. To anyone who knows how to read exponents, this tells us exactly how many moves we require.

Some people might prefer a way of writing the answer that was less exact, but looked more familiar. Well, perhaps we can do that, too. Now, we have seen that  $2^{10} = 1024$ .

We would be approximately correct (our error would be less than 3 percent which is usually considered reasonable) if we said  $2^{10} = 1000 = 10^3$ . Now,

$$\begin{aligned}
 2^{100} &= 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \\
 &= (2^{10})^{10} \\
 &\doteq (10^3)^{10} \\
 &= 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \\
 &= 10^{30} \\
 &= 1,000,000,000,000,000,000,000,000,000,000.
 \end{aligned}$$

From a number as long as this, it would make next to no difference were we to add one,

$$1,000,000,000,000,000,000,000,000,000,001,$$

or to subtract 1, so we shall not bother with the "minus" part of our formula. If we use 100 washers, we shall require approximately 1,000,000,000,000,000,000,000,000,000,000 moves.

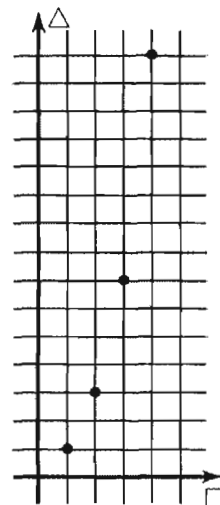
If someone bet you \$100 that you couldn't work the hard three-peg problem starting with 100 washers, would you be wise to accept? You can buy washers for one-tenth cent apiece, so 100 washers would only cost 10 cents. Would you accept the bet?

(17) Can you write a formula for this function?

(18) Can you make a graph for this function?

(17)  $2^n - 1 = \triangle$

(18)



Notice that this graph is not a straight-line graph; this function is what mathematicians call a "nonlinear function."

(19) Larry says that if  $P_n$  is the number of moves you need for  $n$  washers, then

$$P_n = (2 \times P_{n-1}) + 1.$$

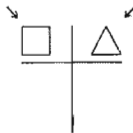
Do you agree?

(19) **Larry is correct.**

### THE METAL-SPRING FUNCTION

(20) Hang up a coiled metal spring, and hang weights on it. You can get a function like this:

Number of grams of weight attached to spring



Number of inches the spring stretches

(20) **You may want to view the Madison Project film entitled "Weights and Springs." It's up to you what model you use. It is usually reasonably accurate to use a straight-line graph. But is it really perfectly accurate?**

(21) Can you make a graph for this function?

(21) and (22) **See the film "Weights and Springs," or simply use careful observation and analysis on your own spring.**

(22) Can you write a formula for this function, using  $\square$ 's and  $\triangle$ 's?

(23) How much would the spring stretch if you hung 27 grams on it?

(23) **This will depend upon your spring.**

(24) How much would the spring stretch if you hung 41 grams on it?

(24) **This will depend upon your spring.**

(25) How much would the spring stretch if you hung 1,000,000 grams on it?

(25) **This will depend upon your spring; it might be badly overloaded long before you reached 1,000,000 grams.**

(26) How much would the spring stretch if you hung 0 grams on it?

(26) **Presumably, it would stretch 0 inches.**

(27) How much would the spring stretch if you hung  $\frac{1}{100}$  gram on it?

(27) **Unless your spring is very delicate, this will be too small to measure. Your guess as to what happens is as good as mine.**

### THE RUBBER-BAND FUNCTION

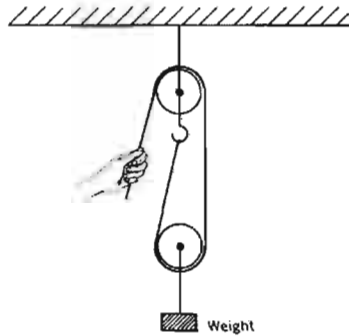
[page 87]

(28) This is exactly like the "metal-spring function," except that you use a "chain" of rubber bands instead of a metal spring. Can you make a table for this function? Can you make a graph for this function? Can you write a formula for this function, using  $\square$ 's and  $\triangle$ 's?

(28) **Usually rubber bands are far more complicated than metal springs. You will probably find that a linear function (that is, a straight-line graph) is not satisfactory in this case. You need a more complicated model.**

THE PULLEY-DISTANCE FUNCTION

(29) Arrange a pulley like this:



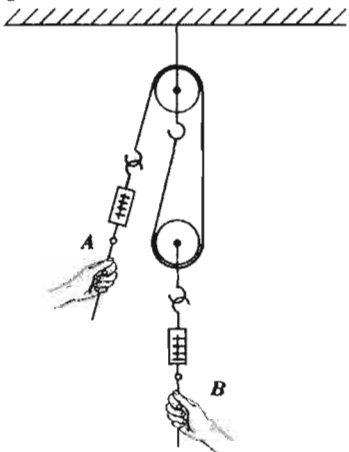
Make a table:

Number of inches hand moves	Number of inches weight moves
□	△

(30) Can you make a graph for this function? Can you write a formula for it, using □'s and △'s?

THE PULLEY-FORCE FUNCTION

(31) Use the same pulley as in questions 29 and 30. Use two spring balances (or some other method for measuring forces), like this:



Pull hard enough at *A* and *B* so that the two forces "just balance," and the rope and pulley wheels don't move. [page 88]

Can you fill in part of this table?

Force at <i>A</i>	Force at <i>B</i>
□	△

Can you make a graph for this function? Can you write a formula for this function?

(29) through (32) See what you can learn from your own laboratory work and careful analysis. Don't worry that things never work out exactly—that's life!

(32) Linda's father says that physicists use **conservation laws**. By looking at your work with the pulley, can you find anything that is **unchanged** (or, as the physicists say, **conserved**)?

(33) Jerry says that the hand at *A* moves farther, but doesn't pull as hard as the force at *B*. Jerry says that the sum

$$F + d$$

is **conserved**, where *F* is the force and *d* is the distance moved.

That is, if *F(A)* and *d(A)* mean the force and distance at *A*, and *F(B)* and *d(B)* mean the force and distance at *B*, then

$$F(A) + d(A) = F(B) + d(B).$$

Do you agree?

(34) Toby has some pieces of wood 1 cm by 1 cm by 5 cm, shaped like a block or a brick:



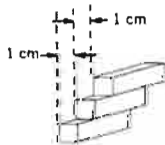
What is the volume of one piece of wood like this? What is the surface area?

(35) Suppose Toby glues together two of the pieces, like this:



What will the volume be? What will the exposed surface area be?

(36) Suppose Toby glues three pieces together, like this:



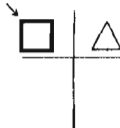
What will the volume be? What will the exposed surface area be?

[page 89]

(37) For Toby's "stairs," can you fill in this table?

Number of blocks of wood

Exposed surface area



(33) Jerry's law does not work, as you should be able to show with the data from your pulley. What law does work?

(34) The volume is 5 cubic centimeters, which might be abbreviated as either 5 cc or 5 cm<sup>3</sup>. The total surface area is 22 square centimeters, or 22 cm<sup>2</sup>.

(35) Volume: 10 cm<sup>3</sup>. Total exposed surface area: 36 cm<sup>2</sup>.

(36) Volume: 15 cm<sup>3</sup>. Total exposed surface area: 50 cm<sup>2</sup>.

(37) Actually, we can make two relevant tables. The table for the volume couldn't be simpler:

Number of blocks of wood

Volume (in cm<sup>3</sup>)

Number of blocks of wood	Volume (in cm <sup>3</sup> )
1	5
2	10
3	15
4	20
⋮	⋮
⋮	⋮

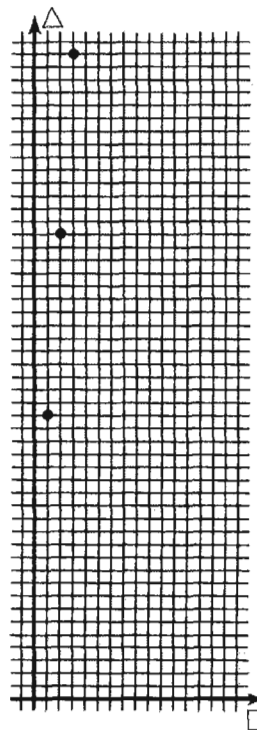
The table for the exposed surface area is a good bit trickier, however:

Number of blocks of wood	□	△
1		22
2		36
3		50
4		64
5		78
⋮		⋮

This problem, originally suggested by Professor David Page of the University of Illinois, has been used with children of various ages for many years, now, and has been written about in several places. See *Discovery, Teachers' Text*, pp. 271-273; also Page (172).

(38) Can you make a graph for this function?

(38)



(39) Can you write a formula for this function?

(39) There are many possibilities (see *Discovery, Teachers' Text*, pp. 271-273). Here are a few:

$$22 + [(\square - 1) \times 14] = \triangle$$

(which can be obtained by noticing that after the first block, each additional block adds 14 cm<sup>2</sup> of additional surface area)

$$(14 \times \square) + 8 = \triangle$$

(which can be obtained from the graph, if we have observed carefully the "slope" and "intercept" patterns on linear graphs)



$$\begin{array}{ccccccc}
 (\square \times 5) & + & (\square \times 5) & + & 10 & + & (\square - 1) & + & (\square - 1) & + & 2 \times \square & = & \triangle \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 \text{Front} & & \text{Back} & & \text{Top} & & \text{Stairs} & & \text{Stairs} & & \text{Both} & & \\
 & & & & \text{and} & & \text{"facing} & & \text{"facing} & & \text{ends} & & \\
 & & & & \text{bottom} & & \text{up"} & & \text{down"} & & & & 
 \end{array}$$

(40) If Toby used 10 blocks, what would the exposed surface area be?

(40) The formula

$$(14 \times \square) + 8 = \triangle$$

is probably the easiest to work with, so we shall use it. (Would the others give the same answers?) Making numerical replacements for the variables:

$$\begin{array}{l}
 \text{UV: } 10 \rightarrow \square \\
 (14 \times 10) + 8 = \square \\
 140 + 8 = \square \\
 148 = \square
 \end{array}$$

Evidently, to get a *true* statement we must put

$$148 \rightarrow \triangle,$$

so we have

$$\begin{array}{r|l}
 \square & \triangle \\
 \hline
 \vdots & \vdots \\
 10 & 148 \\
 \vdots & \vdots \\
 \vdots & \vdots
 \end{array}$$

The answer is 148 cm<sup>2</sup>.

(41) If Toby used 100 blocks, what would the exposed surface area be?

(41) Again using UV,

$$\begin{array}{l}
 \text{UV: } 100 \rightarrow \square \\
 (14 \times 100) + 8 = \triangle \\
 1408 = \triangle
 \end{array}$$

To get a true statement, we must put

$$1408 \rightarrow \triangle,$$

so we have:

$$\begin{array}{r|l}
 \square & \triangle \\
 \hline
 \vdots & \vdots \\
 100 & 1408 \\
 \vdots & \vdots \\
 \vdots & \vdots
 \end{array}$$

The answer is 1408 cm<sup>2</sup>.

(42) If Toby used  $n$  blocks, what would the exposed surface area be?

(42) UV:  $n \rightarrow \square$   
 $(14 \times n) + 8 = \triangle$

The exposed surface area would be  $(14 \times n) + 8$  cm<sup>2</sup>.

## THE NOTATION $f(x)$

As often happens, one of our main tasks here will be to try to avoid confusion. In traditional ninth-grade algebra, we used the device of "writing letters together" to indicate *multiplication*, for example,

$$AB = (A)(B) = A \times B.$$

We now wish to use a somewhat *similar-looking* notation that does *not* indicate multiplication, but refers instead to something quite different.

Once we have had the *experience* of using "rules," as in

$$(3 \times \square) + 7 = \triangle,$$

or in studying the three-peg game, or in "guessing functions," etc., we can, hopefully, *think about the general process of using some rule or other*. Now, at the third stage in Bruner's Trilogy, we seek a notation for *writing* this idea of "using some rule or other."

Suppose someone has made up a rule. We tell them 3, and they answer 7. We tell them 4, and they answer 9.

We tell them	□	△	They answer
	3	7	
	4	9	

The new notation we are seeking is sometimes written this way:

Written	Read
$f(3) = 7$	"f of three equals seven."
$f(4) = 9$	"f of four equals nine."

Alternatively, it is sometimes written this way:

$$\begin{aligned} 3 &\rightarrow 7 \\ 4 &\rightarrow 9 \end{aligned}$$

A third choice is to write:

Written	Read
$F: 3 \rightarrow 7$	"The rule $F$ maps 3 into 7."
$F: 4 \rightarrow 9$	"The rule $F$ maps 4 into 9."

Using the first of these methods, we could write the general case for the rule as:

Written	Read
$f(\square) = (2 \times \square) + 1$	"f of box equals two times box plus one."

Using the third method, we could write:

Written	Read
$F: \square \rightarrow (2 \times \square) + 1$	"The rule $F$ maps box into two times box plus one."

Notice that  $f(3)$  does *not* mean "multiply  $f$  by 3."



## CHAPTER 28

The Notation  $f(x)$ 

[page 90]

Can you find the truth set for each open sentence?

(1)  $(\square \times \square) - (15 \times \square) + 26 = 0$

(2)  $(\square \times \square) - (-5 \times \square) + *6 = 0$

(3)  $(\square \times \square) - (*2 \times \square) + *15 = 0$

(4)  $(\square \times \square) - (-3 \times \square) + *4 = 0$

(5)  $(\square \times \square) - (-3 \times \square) + *10 = 0$

(6)  $(\square \times \square) - (-9 \times \square) + *20 = 0$

(7) Pam made up a rule, which worked like this

If you told her 2, she answered 5.

If you told her 3, she answered 6.

If you told her 4, she answered 7.

Debbie wrote the rule this way:

$P: 2 \rightarrow 5$

$P: 3 \rightarrow 6$

$P: 4 \rightarrow 7$

$P: \square \rightarrow \square + 3$

Can you complete these?

(a)  $P: 5 \rightarrow \underline{\quad}$

(b)  $P: 6 \rightarrow \underline{\quad}$

(c)  $P: \underline{\quad} \rightarrow 15$

(d)  $P: x \rightarrow \underline{\quad}$

(e)  $P: w \rightarrow \underline{\quad}$

(f)  $P: 1.002 \rightarrow \underline{\quad}$

(g)  $P: x + 3 \rightarrow \underline{\quad}$

(1)  $\{13, 2\}$

(2)  $\{-2, -3\}$

(3)  $\{*5, -3\}$

(4)  $\{-4, *1\}$

(5)  $\{-5, *2\}$

(6)  $\{-4, *5\}$

Now we start to work with "functions."

(7) (a)  $P: 5 \rightarrow 8$

(b)  $P: 6 \rightarrow 9$

(c)  $P: 12 \rightarrow 15$

(d)  $P: x \rightarrow x + 3$

(e)  $P: w \rightarrow w + 3$

(f)  $P: 1.002 \rightarrow 4.002$

(g)  $P: x + 3 \rightarrow x + 6$

## ANSWERS AND COMMENTS

These first six problems are "warm-up" questions to help the students to recall the "secrets" for quadratic equations (see Chapter 10) before we begin our new work for this lesson, which will, of course, deal with the notation  $f(x)$ .

(8) Frank used Pam's rule this way:

He wrote:

$$P(2) = 5$$

$$P(3) = 6$$

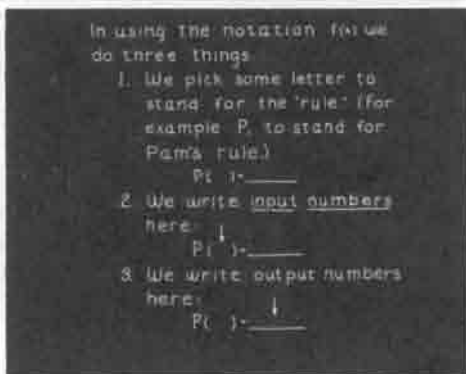
$$P(\square) = \square + 3$$

He read it as:

"P of 2 equals 5."

"P of 3 equals 6."

Frank wrote this on the chalkboard to explain how he wrote Pam's rule:



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Can you fill in the missing input or output numbers?

- (a)  $P(4) = \underline{\quad}$
- (b)  $P(1) = \underline{\quad}$
- (c)  $P(\frac{1}{2}) = \underline{\quad}$
- (d)  $P(\underline{\quad}) = 22$
- (e)  $P(x) = \underline{\quad}$
- (f)  $P(t) = \underline{\quad}$
- (g)  $P(w) = \underline{\quad}$
- (h)  $P(\underline{\quad}) = s + 5$
- (i)  $P(y + 4) = \underline{\quad}$

(9) John made the rule

$$J: x \rightarrow x^2 - 2x + 3.$$

Can you complete these?

- (a)  $J: 3 \rightarrow \underline{\quad}$
- (b)  $J: 4 \rightarrow \underline{\quad}$
- (c)  $J: \underline{\quad} \rightarrow 18$
- (d)  $J: w \rightarrow \underline{\quad}$
- (e)  $J: s + 3 \rightarrow \underline{\quad}$

- (8) (a)  $P(4) = 7$  (Read: "P of 4 equals 7.")
- (b)  $P(1) = 4$  (Read: "P of 1 equals 4.")
- (c)  $P(\frac{1}{2}) = 3\frac{1}{2}$  (Read: "P of  $\frac{1}{2}$  equals  $3\frac{1}{2}$ .")
- (d)  $P(19) = 22$  (Read: "P of 19 equals 22.")
- (e)  $P(x) = x + 3$  (Read: "P of x equals  $x + 3$ .")
- (f)  $P(t) = t + 3$
- (g)  $P(w) = w + 3$
- (h)  $P(s + 2) = s + 5$  (Read: "P of  $s + 2$  equals  $s + 5$ .")
- (i)  $P(y + 4) = y + 7$

- (9) (a)  $J: 3 \rightarrow 6$
- (b)  $J: 4 \rightarrow 11$
- (c)  $J: 5 \rightarrow 18$ , or  $J: 3 \rightarrow 18$

Part (c) calls for a little writing. If  $J: r \rightarrow 18$ , then this means

$$r^2 - 2r + 3 = 18.$$

Subtracting 18 from each side, we get

$$r^2 - 2r + 15 = 0,$$

a quadratic equation for which the truth set is  $\{5, 3\}$ . Hence, we have  $J: 5 \rightarrow 18$  and also  $J: 3 \rightarrow 18$ .

- (d)  $J: w \rightarrow w^2 - 2w + 3$
- (e)  $J: s + 3 \rightarrow (s + 3)^2 - 2(s + 3) + 3$ ,  
or  $J: s + 3 \rightarrow s^2 + 4s + 6$

(10) We can also use Frank's method to write John's rule:

$$J(x) = x^2 - 2x + 3.$$

Can you complete these?

- (a)  $J(0) = \dots$
- (b)  $J(1) = \dots$
- (c)  $J(2) = \dots$
- (d)  $J(100) = \dots$
- (e)  $J(\dots) = 83$
- (f)  $J(t + 2) = \dots$
- (g)  $J(2 \times w) = \dots$
- (h)  $J(\square) = \dots$
- (i)  $J(N) = \dots$

(11) Ruth made up the rule

$$R(\square) = (\square \times \square) - (7 \times \square) + 10.$$

Can you find the truth set for the open sentence

$$R(\square) = 0?$$

(12) For Ruth's rule

$$R: \square \rightarrow [(\square \times \square) - (7 \times \square) + 10].$$

Alex made up the open sentence

$$R(\square) = 2.$$

Can you find its truth set?

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(13) Al used Ruth's rule and made up the open sentence

$$R(\square) = 4.$$

Can you find the truth set for Al's open sentence?

(14) Can you find  $R(7)$ ?

(15) Can you find  $R(-1)$ ?

- (10) (a)  $J(0) = 3$
- (b)  $J(1) = 2$
- (c)  $J(2) = 3$
- (d)  $J(100) = 10,000 - 200 + 3 = 9803$
- (e)  $J(10) = 83$ , or  $J(8) = 83$

This, too, calls for a little writing. If  $J(s) = 83$ , then we have

$$s^2 - 2s + 3 = 83,$$

or

$$s^2 - 2s + -80 = 0,$$

with the truth set  $\{+10, -8\}$ . Hence we have  $J(10) = 83$  and also  $J(-8) = 83$ .

$$(f) J(t + 2) = (t + 2)^2 - 2(t + 2) + 3 = t^2 + 2t + 3$$

$$(g) J(2 \times w) = (2 \times w)^2 - 2(2 \times w) + 3 = 4w^2 - 4w + 3$$

$$(h) J(\square) = (\square \times \square) - (2 \times \square) + 3$$

$$(i) J(N) = N^2 - 2N + 3$$

$$(11) (\square \times \square) - (7 \times \square) + 10 = 0; \{5, 2\}$$

$$(12) (\square \times \square) - (7 \times \square) + 10 = 2;$$

adding 2 to each side, we get

$$(\square \times \square) - (7 \times \square) + 12 = 0; \{4, 3\}$$

$$(13) (\square \times \square) - (7 \times \square) + 10 = 4;$$

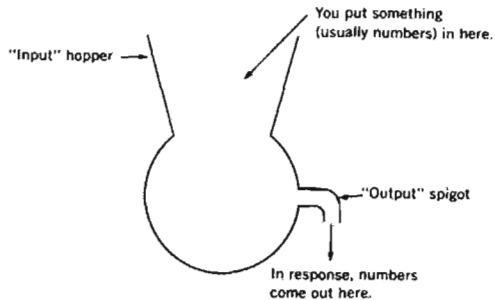
subtracting 4 from each side, we get

$$(\square \times \square) - (7 \times \square) + 6 = 0; \{6, 1\}$$

$$(14) R(7) = 10 \quad (\text{Read: "R of 7 equals 10."})$$

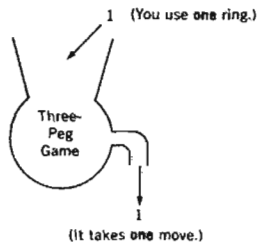
$$(15) R(-1) = 1 + 7 + 10 = 18$$

(16) Ruth's father says the idea of a "function" or a "rule" is always something like this:

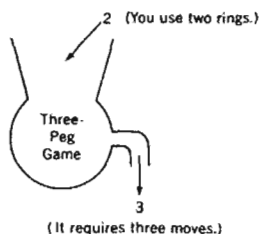


What do you think?

(17) Charles says the three-peg game is a good example of a function. If we use one ring, it takes one move to complete the puzzle:



If, instead, we use two rings, the puzzle requires three moves:



Suppose we construct a "function machine" that will match up numbers the same way that the three-peg game does. What number would come out of the spigot of this machine if you tossed "three" into the hopper?

(18) How many different ways of looking at functions do you think there are?

(16) This question is meant to focus the student's attention on the input-output aspect of functions.

(17) It would take 5 moves to complete the puzzle. Here the student is to recognize the pattern produced in the three-peg game; if you use  $n$  rings it takes  $2n - 1$  moves to complete the puzzle.

(18) Functions have been described using tables, graphs, and equations. They have also been described as mappings,

$$n \rightarrow n + 3;$$

as rules in sentence form,

If I tell you  $n$ , you tell me  $n + 3$ ;

and as rules written in function notation,

$$f(n) = n + 3.$$

Finally, functions have been described in terms of input and output numbers.

Perhaps your pupils will be able to describe functions in other ways.

## SOME OPERATIONS ON EQUATIONS



### CHAPTER 29

## Some Operations on Equations

[page 93]

(1) Jerry says that two different equations may have the same truth set. What do you suppose he means?

(2) Do these equations have the same truth set?

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$$(\square - 2) \times (\square - 3) = 0$$

(3) Do these equations have the same truth set?

$$(\square \times \square) - (4 \times \square) + 4 = 0$$

$$3 + \square = 5$$

(4) Do these open sentences have the same truth set? (Use only integers as replacements for the variable  $\square$ .)

$$3 < \square + 1 < 6$$

$$(\square \times \square) - (7 \times \square) + 12 = 0$$

(5) Do these open sentences have the same truth set?

$$2 < \square + 1 < 5$$

$$(\square \times \square) - (7 \times \square) + 10 = 0$$

## ANSWERS AND COMMENTS

(1) This probably is clear. As an example,

$$(2 \times \square) + 3 = 11$$

and

$$\square + 10 = 14$$

are two different equations, but either has the truth set  $\{4\}$ .

(2) Yes. Each has the truth set  $\{2, 3\}$ .

(3) Yes. Each has the truth set  $\{2\}$ .

(4) Well, let's try it out and see. For

$$3 < \square + 1 < 6$$

the truth set (using only integers) is  $\{3, 4\}$ .

Now, let's look at

$$(\square \times \square) - (7 \times \square) + 12 = 0.$$

It evidently has the truth set  $\{3, 4\}$ .

Hence, both open sentences *do* have the same truth set.

(5) Let's try it out. The inequality

$$2 < \square + 1 < 5$$

has the truth set  $\{2, 3\}$ .

The quadratic equation

$$(\square \times \square) - (7 \times \square) + 10 = 0$$

has the truth set  $\{2, 5\}$ .

Evidently, these two open sentences *have different truth sets*.

(6) Eileen says that two equations which have the same truth set are called **equivalent equations**. What do you think?

(6) **Eileen is correct.**

For each of the following pairs of equations, can you decide whether or not the two equations are equivalent?

$$(7) \begin{cases} (2 \times \square) + 1 = 177 \\ (2 \times \square) + 2 = 178 \end{cases}$$

(7) **Equivalent**

$$(8) \begin{cases} \square + 3 = 10 \\ \square + 5 = 12 \end{cases}$$

(8) **Equivalent**

$$(9) \begin{cases} (2 \times \square) + 11 = 34 \\ (2 \times \square) + 12 = 35 \end{cases}$$

(9) **Equivalent**

$$(10) \begin{cases} (3 \times \square) + 191 = 273 \\ (3 \times \square) + 196 = 278 \end{cases}$$

(10) **Equivalent**

$$(11) \begin{cases} \square + 3 = 7 \\ (2 \times \square) + 6 = 14 \end{cases}$$

(11) **Equivalent**

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$$(12) \begin{cases} \square + 5 = 21 \\ \square + 10 = 42 \end{cases}$$

(12) **Not equivalent**

$$(13) \begin{cases} (3 \times \square) + 11 = 51 \\ (3 \times \square) + 22 = 102 \end{cases}$$

(13) **Not equivalent**

$$(14) \begin{cases} (3 \times \square) + 2 = 100 \\ (5 \times \square) + 4 = 200 \end{cases}$$

(14) **Not equivalent**

$$(15) \begin{cases} (3 \times \square) + 2 = 100 \\ (6 \times \square) + 4 = 200 \end{cases}$$

(15) **Equivalent**

$$(16) \begin{cases} (3 \times \square) + 2 = 100 \\ (6 \times \square) + 2 = 200 \end{cases}$$

(16) **Not equivalent**

$$(17) \begin{cases} (3 \times \square) + 2 = 100 \\ (3 \times \square) + 102 = 200 \end{cases}$$

(17) **Equivalent**



$$(18) \begin{cases} \square + 3 = 10 \\ (2 \times \square) + 6 = 20 \end{cases}$$

(18) Equivalent

$$(19) \begin{cases} (13 \times \square) + 1791 = 2564 \\ (15 \times \square) + 1792 = 2565 \end{cases}$$

(19) Not equivalent

(20) There are certain things you can do to an equation to produce a new equation. For example, you might add 3 to the "left-hand side" of the equation.

If you start with the equation

$$\square + 10 = 100$$

and add 3 to the left-hand side, you get the new equation

$$\square + 13 = 100.$$

If you add 3 to the left-hand side of one equation, in order to get a new equation, do you suppose the new equation will have the same truth set as the original equation?

(21) Do something to the equation

$$(3 \times \square) + 2 = 35$$

so as to produce a new equation. Did you change the truth set?

(20) No, in general, it will not.

(21) The discussion here will depend upon your class.

(22) Do something to the equation

$$\square + 3 = 7$$

so as to produce a new equation which will have a different truth set.

(22) This will depend upon your class.

(23) Do something to the equation

$$\square + 3 = 7$$

so as to produce a new but equivalent equation.

(23) This will depend upon your class.

(24) Beryl says that things you do to equations that produce new equations with the same truth set are called transform operations. What do you think?

(24) Beryl is correct.

(25) Can you use a transform operation on the equation

$$(3 \times \square) + 25 = 85?$$

What new equation did you get?

(25) This will depend upon your class.

(26) Lex says that he knows five different kinds of transform operations. How many do you know?

(26) (i) Adding the same number to each side of an equation

(ii) Subtracting the same number from each side of an equation

(iii) Multiplying each side of an equation by the same number, provided that number is not zero

(iv) Dividing each side of an equation by the same number, provided that the number is not zero

(v) Using PN, with any *identity* or *true statement*; for example,

$$2 \times \square = 8$$

can be changed to

$$\square + \square = 8.$$

(27) Jerry said he thought Lex was wrong. Lex said, "Let me give you a clue!" Then Lex wrote:

$$\left\{ \begin{array}{l} \square + 3 = 5 \\ \square + 103 = 105 \end{array} \right.$$

$$\left\{ \begin{array}{l} \square + 3 = 5 \\ (2 \times \square) + 6 = 10 \end{array} \right.$$

$$\left\{ \begin{array}{l} \square + 3 = 5 \\ (2 \times \square) + 6 = 10 \end{array} \right.$$

$$\left\{ \begin{array}{l} \square + 3 = 5 \\ \square + 2 = 4 \end{array} \right.$$

$$\left\{ \begin{array}{l} \square + 3 = 5 \\ \square + 2 = 4 \end{array} \right.$$

$$\left\{ \begin{array}{l} (2 \times \square) + 4 = 16 \\ (1 \times \square) + 2 = 8 \end{array} \right.$$

$$\left\{ \begin{array}{l} (2 \times \square) + 4 = 16 \\ (1 \times \square) + 2 = 8 \end{array} \right.$$

$$\left\{ \begin{array}{l} 2 \times \square = 102 \\ \square + \square = 102 \end{array} \right.$$

$$\left\{ \begin{array}{l} 2 \times \square = 102 \\ \square + \square = 102 \end{array} \right.$$

Do you know what Lex meant?

(28) Jeannie started with nine equations and did "something" to each one, to produce a new equation in each case. Can you describe what Jeannie did in each case? Was it a **transform operation** or not?

(a)  $\left\{ \begin{array}{l} \text{Jeannie's original equation: } 3 + \square = 21 \\ \text{Jeannie's new equation: } 4 + \square = 21 \end{array} \right.$

(b)  $\left\{ \begin{array}{l} \text{Jeannie's original equation: } 3 + \square = 50 \\ \text{Jeannie's new equation: } 4 + \square = 55 \end{array} \right.$

(c)  $\left\{ \begin{array}{l} \text{Jeannie's original equation: } (2 \times \square) + 5 = 10 \\ \text{Jeannie's new equation: } (4 \times \square) + 10 = 20 \end{array} \right.$

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(a)  $\left\{ \begin{array}{l} \text{Jeannie's original equation: } 3 + \square = 21 \\ \text{Jeannie's new equation: } 4 + \square = 21 \end{array} \right.$

$$(2 \times \square) + 5 = 10$$

Jeannie's new equation:

$$(4 \times \square) + 10 = 20$$

**Adding the same number to each side of the equation**

**Multiplying each side by the same (non-zero) number**

**Subtracting the same number from each side of the equation**

**Dividing each side by the same (non-zero) number**

**Using PN, plus an identity or a true statement**  
(Here, the identity  $2 \times \square = \square + \square$  has been used.)

(28) (a) She added 1 to the left-hand side, but left the right-hand side unchanged. **Not** a transform operation.

(b) She added 1 to the left-hand side and added five to the right-hand side. **Not** a transform operation.

(c) She multiplied each side of the equation by 2. This *is* a transform operation; the truth set was not changed by this change in the equation.

(d) Jeannie's original equation:  
 $(\square + \square) + 5 = \square + 8$   
 Jeannie's new equation:  
 $(2 \times \square) + 5 = \square + 8$

(e) Jeannie's original equation:  
 $(\square + \square) + 10 = 15$   
 Jeannie's new equation:  
 $\square + \square = 5$

(f) Jeannie's original equation:  
 $(\square + \square) + 9 = 49 + \square$   
 Jeannie's new equation:  
 $\square + 9 = 49$

(g) Jeannie's original equation:  
 $(2 \times \square) + 7 = 31$   
 Jeannie's new equation:  
 $(2 \times \square) + 14 = 62$

(h) Jeannie's original equation:  
 $(\square + 3) = 10$   
 Jeannie's new equation:  
 $(\square + 3) + 5 = 10 + 5$

(i) Jeannie's original equation:  
 $(\square \times \square) - (9 \times \square) + 14 = 0$   
 Jeannie's new equation:  
 $(\square - 2) \times (\square - 7) = 0$

(29) What do we mean by a transform operation?

(d) She used the identity

$$\square + \square = 2 \times \square,$$

plus PN. This is a transform operation.

(e) She subtracted 10 from each side of the equation. This is a transform operation. (That is to say, the truth sets of the two equations are necessarily the same.)

(f) She subtracted  $\square$  from each side of the equation. This is a transform operation.

(g) She multiplied *some* terms by 2, but left others unchanged. This is *not* a transform operation.

(h) She added 5 to each side of the equation. This is a transform operation.

(i) This is a tricky one, but an important one. Jeannie used the identity

$$(\square \times \square) - (9 \times \square) + 14 = (\square - 2) \times (\square - 7),$$

plus PN. This is a transform operation.

(29) Actually, there *are* some subtleties here that may deserve discussion, but we ordinarily omit them at this stage. The children don't think of them, and we prefer not to introduce them at this time. Hence, we settle for the simple statement:

A transform operation is something you can do to an equation that will leave the truth set unchanged.

Or, on this same level of simplification:

A transform operation is a systematic procedure for starting with one equation and obtaining a new, but *equivalent*, equation.

(30) Are these equivalent equations?

$$\begin{aligned}(\square \times \square) - 16 &= 0 \\ [(\square \times \square) - 16] + 25 &= 25\end{aligned}$$

(31) John says that Lex's "five kinds of transform operations" are not really all different. What do you think?

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(32) John wrote:

$$7 - 5 = 7 + ^{-}5$$

What do you suppose he meant?

(33) John also wrote:

$$\begin{aligned}(3 \times \square) + 10 &= 25 \\ [(3 \times \square) + 10] - 10 &= 25 - 10 \\ [(3 \times \square) + 10] + ^{-}10 &= 25 + ^{-}10\end{aligned}$$

What do you suppose he meant?

(34) How many different kinds of transform operations do you know?

Can you find the truth set for each equation?

(35)  $(\square \times \square) - (5 \times \square) + 10 = 4$

(36)  $(\square \times \square) - (15 \times \square) + 5 = ^{-}31$

(37)  $\begin{aligned}(\square \times \square) - (12 \times \square) + 20 \\ = ^{-}2 + (1 \times \square)\end{aligned}$

(38)  $\begin{aligned}[(\square + \square) \times (\square + 13)] + \square \\ = [(\square + \square) \times (\square + 13)] + 7\end{aligned}$

(39)  $(3 \times \square) + 1951 = (2 \times \square) + 1500$

(30) Yes.

(31) John is thinking (correctly, to be sure) that "subtracting a number" is really the same thing as adding the "opposite" of the number. Since we allow both positive and negative numbers, we can include "subtraction" *within the notion of addition*. The same holds for multiplication and division.

(32) The "subtraction" problem  $7 - 5$  is the same as the "addition" problem  $7 + ^{-}5$ .

(33) Again, "subtracting" 10 from each side is really not different from "adding"  $^{-}10$  to each side. Hence, on Lex's list of five kinds of transform operations, we can delete two (namely, those that speak of "subtracting" and "dividing").

(34) If you combine "addition" and "subtraction," and if you also combine "multiplication" and "division," then you would presumably list three "different" kinds of transform operations:

- (i) Adding the same number to each side of an equation
- (ii) Multiplying both sides by the same non-zero number
- (iii) Using PN plus some identity or true statement

(35) One method for solving this problem would be to *subtract 4* from each side of the equation, to get

$$(\square \times \square) - (5 \times \square) + 6 = 0; \{2, 3\}.$$

(36) Similarly, you can get the equivalent equation

$$(\square \times \square) - (15 \times \square) + 36 = 0; \{12, 3\}.$$

(37) Subtract

$$^{-}2 + (1 \times \square)$$

from each side, to get the equivalent equation

$$(\square \times \square) - (13 \times \square) + 22 = 0; \{2, 11\}.$$

(38)  $\{7\}$  (How did we do it so easily?)

(39)  $\{-451\}$  (Can you do it without writing?)

(40) Are these equivalent equations?

$$(\square - 3) = 5$$

$$(\square - 3)^2 = 25$$

(40) No. For the equation

$$\square - 3 = 5,$$

the truth set is  $\{8\}$ .

However, for the equation

$$(\square - 3)^2 = 25,$$

we can say that *either*

$$\square - 3 = 5 \quad \text{or} \quad \square - 3 = -5.$$

Hence, the truth set for

$$(\square - 3)^2 = 25$$

is  $\{8, -2\}$ . Since  $\{8\} \neq \{8, -2\}$ , the equations are *not* equivalent.



## SOME OPERATIONS ON INEQUALITIES

This short chapter is intended to make sure that the students notice that some of the transform operations on equations do not work if they are applied to inequalities. For further reading, see Appendix A, Dupree (72).



### CHAPTER 30

## Some Operations on Inequalities

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When mathematicians say that two inequalities are "equivalent," they mean that both inequalities have the same truth set.

In this chapter, let's agree to use only positive integers as replacements for the variables.

- (1) David made up the inequality

$$(2 \times \square) + 1 < 10.$$

Can you make up an equivalent inequality?

- (2) Tom made up the inequality

$$\square + 5 < 8.$$

Can you make up an equivalent inequality?

## ANSWERS AND COMMENTS

- (1) Since we are using only positive integers as replacements for the variables, the inequality

$$(2 \times \square) + 1 < 10$$

has the truth set  $\{1, 2, 3, 4\}$ .

Now, an "equivalent inequality" would be one having this same truth set. Hence (to give one possible example)

$$8 < \square + 8 < 13$$

would be an equivalent inequality.

- (2) Here are several possibilities:

$$\square + 6 < 9$$

$$\square + 7 < 10$$

$$\square + 8 < 11$$

$$(2 \times \square) + 10 < 16$$

$$\square + 4 < 7$$

$$\square + 3 < 6$$

$$\square + 2 < 5$$

$$\square + 1 < 4$$

$$\square < 3$$

(3) Are these inequalities equivalent?

$$\square + 3 < 10$$

$$(2 \times \square) + 6 < 20$$

(4) Are these inequalities equivalent?

$$\square - 2 < 4$$

$$-1 \times (\square - 2) < -4$$

(5) Are these inequalities equivalent?

$$\square + 3 < 7$$

$$\square + 4 < 10$$

(6) What transform operations can you find for inequalities?

(3) The inequality

$$\square + 3 < 10$$

has the truth set  $\{1, 2, 3, 4, 5, 6\}$ .

The inequality

$$(2 \times \square) + 6 < 20$$

has the truth set  $\{1, 2, 3, 4, 5, 6\}$ . Since the truth sets *are* the same, the inequalities are *equivalent*.

(4) The inequality

$$\square - 2 < 4$$

has the truth set  $T_1 = \{1, 2, 3, 4, 5\}$ .

The inequality

$$-1 \times (\square - 2) < -4$$

requires a bit more thought. Let's just try a few numbers "at random" to see what happens:

$$\text{UV: } 1 \longrightarrow \square$$

$$-1 \times (1 - 2) < -4$$

$$-1 \times -1 < -4$$

$$-1 < -4 \quad \text{False}$$

Thus 1 is not an element of the truth set,  $T_2$ , for this second inequality.

Since  $1 \in T_1$  and  $1 \notin T_2$ , we know immediately that  $T_1 \neq T_2$  and that the inequalities are *not equivalent*.

This example shows us that multiplying both sides of an inequality by  $-1$  does not constitute a transform operation.

(5) The inequality

$$\square + 3 < 7$$

has the truth set  $T_1 = \{1, 2, 3\}$ . The inequality

$$\square + 4 < 10$$

has the truth set  $T_2 = \{1, 2, 3, 4, 5\}$ . The inequalities are *not equivalent*.

(6) (i) **Adding the same number to both sides**

(ii) **Subtracting the same number from each side (if you choose to consider this different from i)**

(iii) **Multiplying both sides by the same *positive* number**

(iv) **Multiplying both sides by the same *negative* number, and at the same time replacing the symbol  $<$  by  $>$ , or vice versa**

(v) **Using any identity or true statement and PN**

## “VARIABLES” VS. “CONSTANTS”

We have been *using* this distinction for some time now. In the present chapter we try to make it more explicit. Perhaps our best approach is by way of an example.

**Example 1.** A teacher wants to write an examination for his students, with the following five questions:

Examination

Find the truth set for each open sentence.

1.  $(\square \times \square) - (5 \times \square) + 6 = 0$
2.  $(\square \times \square) - (3 \times \square) + 2 = 0$
3.  $(\square \times \square) - (15 \times \square) + 26 = 0$
4.  $(\square \times \square) - (21 \times \square) + 20 = 0$
5.  $(\square \times \square) - (50 \times \square) + 96 = 0$

The teacher wants to make a brief memo to himself, indicating all five questions with as little writing as possible. He could write:

$$(\square \times \square) - (a \times \square) + b = 0$$

1. UV:  $5 \rightarrow a, 6 \rightarrow b$
2. UV:  $3 \rightarrow a, 2 \rightarrow b$
3. UV:  $15 \rightarrow a, 26 \rightarrow b$
4. UV:  $21 \rightarrow a, 20 \rightarrow b$
5. UV:  $50 \rightarrow a, 96 \rightarrow b$

In old-fashioned (but valuable) language, the symbol  $\square$  represents a “variable” or an “unknown,” and the symbols  $a$  and  $b$  represent “constants.”

**Example 2:** If an equation  $f(x) = y$  leads to a *straight-line graph*, we have seen earlier in this book (and also in *Discovery*) that the equation can be written in the form

$$(\square \times \text{---}) + \text{---} = \triangle$$

↑
↑  
 Some definite number here      Some definite number here

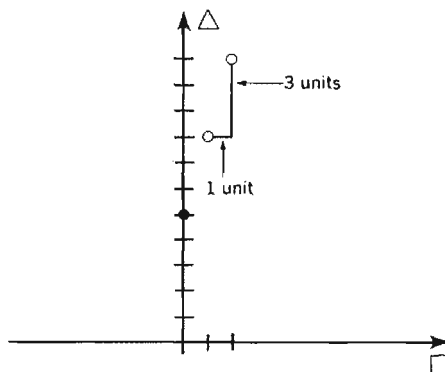
For instance, if the “slope” pattern is “over one unit to the right, and up three units,” then the first missing number must, in fact, be a 3:

$$(\square \times 3) + \text{---} = \triangle.$$

If, further, this line intersects the  $\triangle$  (or vertical) axis at the point (0, 5), then the other missing number must be 5:

$$(\square \times 3) + 5 = \triangle.$$





The "blanks" or "missing numbers"

$$(\square \times \frac{\quad}{\quad}) + \frac{\quad}{\quad} = \triangle$$

represent *constants*, whereas the  $\square$  and  $\triangle$  represent variables. If we use the notation introduced by Viète, and modified by Descartes, we would write

$$(\square \times a) + b = \triangle$$

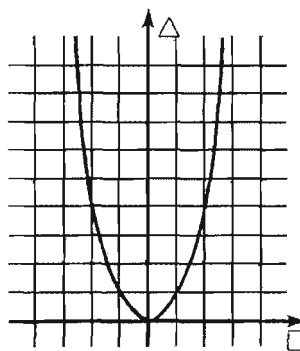
or, alternatively,  $ax + b = y$ .

Now, it is important here that *during the period of time while we are working on a single problem*, the numbers used as replacements for  $a$  and  $b$  *must not be changed*. During the course of working a single problem, we will ordinarily use various different numbers as replacements for the variables  $\square$  and  $\triangle$ . (Geometrically, this corresponds to the fact that we keep the line fixed, and do not move it, but we *do* turn our attention to a *variety of different points on this line*. The constants  $a$  and  $b$  determine which *line* we are talking about; the variables  $\square$  and  $\triangle$  determine which *point* we are talking about.)

Let us give an example of *what not to do*. If, say, we allowed the  $a$  number to vary during a problem, we might try *putting the same number in the  $\square$  and also in  $a$* , so that the equation would become (for simplicity, let's use UV:  $0 \rightarrow b$ )

$$\square \times \square = \triangle,$$

but this is the equation of a parabola



and *not* the equation of a straight line at all. By allowing our "constant" to vary during the discussion of a single problem, we have arrived at nonsense.

So—even if it is difficult to *describe* the difference—the fact remains that "variables" and "constants" *really are different*.

Probably in most cases your intuition and your own good sense are the safest guides in distinguishing the different roles assigned to *variables* (like  $\square$  in the examples just given) and to *constants* (like the  $a$  in the problem above). Don't worry too much about this distinction; most people seem to use variables and constants correctly most of the time, even if they are unable to describe exactly what the difference is.

In many cases, you can imagine that the person who makes up the problem will assign numbers to the "constants"; the person who solves the problem will assign numbers to the variables, as the occasion may demand.

Let's see what we can learn from going through the *Student Discussion Guide*.



## CHAPTER 31

## "Variables" vs. "Constants"

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In the seventeenth century René Descartes, whose work we have encountered earlier, decided to distinguish between different ways that we use  $\square$ ,  $\triangle$ ,  $x$ ,  $y$ ,  $A$ ,  $B$ , and so on. In some cases, Descartes would say that we were dealing with constants. In other cases, he would say we were dealing with variables.\*

In order not to get mixed up, Descartes decided to use letters near the beginning of the alphabet to represent constants. (We shall do the same thing, but we shall add also letters near the beginning of the Greek alphabet; so we shall use  $a$ ,  $b$ ,  $c$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and so forth, to refer to constants.)

When he wanted to refer to what he called variables, Descartes used letters near the end of the alphabet. (We shall too, but we shall also use  $\square$ ,  $\triangle$ , etc. Hence, when we want to refer to what Descartes called variables, we can write:  $x$ ,  $y$ ,  $z$ ,  $w$ ,  $u$ ,  $v$ ,  $\square$ ,  $\triangle$ ,  $\nabla$ ,  $\square$ , etc.)

Modern mathematicians and logicians do not always use the words "variable" and "constant" in quite the same way that Descartes did, but nonetheless Descartes' idea is really still valuable and is still used in one form or another.

\*In fact, Descartes was not the earliest mathematician to decide to distinguish "variables," or "unknowns," from "constants." This distinction was used earlier by the great mathematician François Viète (1540-1603), and also by the Englishman Thomas Harriot (1560-1621) [Incidentally, Sir Walter Raleigh sent Harriot to (what is now) the United States to survey (what is now) North Carolina.] Viète and Harriot wrote the distinction differently, however; they used vowels for "unknowns," and used consonants to represent "constants."

## ANSWERS AND COMMENTS

Now, what was it that Descartes meant, anyhow?  
Let's give some examples first.

**Example 1**

Some airplanes have two engines, some have three, and some have four. Perhaps, then, to the engineer or designer who sets out to design a new airplane, the number of engines is a "variable."

However, to the pilot who flies the plane after it is built, the number of engines is a "constant"—if he takes off with a two-engined plane, he cannot simply decide in midflight to change to three engines.

**Example 2**

Suppose a teacher is making up a test. If he wants to put in one question about quadratic equations, he can write:

Find the truth set for

$$(\square \times \square) - (5 \times \square) + 6 = 0.$$

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Or, if he prefers, he can write:

Find the truth set for

$$(\square \times \square) - (16 \times \square) + 55 = 0.$$

Or he can, instead, write:

Find the truth set for

$$(\square \times \square) - (20 \times \square) + 96 = 0.$$

Hence, to the teacher who is making up the problem, the numbers to go here

$$(\square \times \square) - (\underline{\quad} \times \square) + \underline{\quad} = 0$$

and here

$$(\square \times \square) - (\underline{\quad} \times \square) + \underline{\quad} = 0$$

are "variables."

For the student taking the test, however, these numbers are constants. The student is supposed to answer the question that was actually asked, and not some other question that might have been asked.

Perhaps the important idea is that, while all these letters ( $a$ ,  $b$ ,  $x$ ,  $y$ , etc.) and frames ( $\square$ ,  $\triangle$ , etc.) really name variables (and that is what the modern logician would say), there may be a definite point in time when we choose to make numerical replacements for these variables, and that time may come sooner for some variables than it will for others.

Thus the pilot is still free to determine the direction of the airplane—for him, that is still a variable—but the choice of how many engines to put on it was made long before he took off.

To emphasize this distinction, we could write our quadratic equation problem like this

$$(\square \times \square) - (a \times \square) + b = 0$$

or else like this

$$x^2 - ax + b = 0.$$

To the modern logician,  $\square$ ,  $x$ ,  $a$ , and  $b$  all denote variables. But the teacher will make a numerical replacement for the variables  $a$  and  $b$ , so that—if we are students—by the time the problem gets to us,  $a$  and  $b$  will be constants.

(1) Suppose you are the teacher. Make numerical replacements for the variables  $a$  and  $b$  in the equation

$$(\square \times \square) - (a \times \square) + b = 0,$$

[page 101]

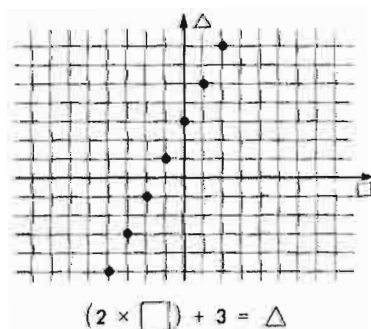
so as to get a reasonably easy examination question. (Make sure you can solve it yourself!)

(2) Now suppose you are the student. Exchange your paper with someone else, and see if you can solve the quadratic equation he made up. (Also, see if he can solve yours!)

(3) In modern language, we would call the  $\square$ ,  $a$ , and  $b$  of question 1 variables. Descartes, however, would have called some of them variables and some of them constants.

Which would Descartes call constants? Which would he call variables?

(4) If we study these graphs, we notice two important patterns.



(1) This will depend upon your class. Here are a few possibilities:

(a) UV:  $7 \rightarrow a, 12 \rightarrow b$

$$(\square \times \square) - (7 \times \square) + 12 = 0; \quad \{3, 4\}$$

(b) UV:  $1968 \rightarrow a, 1967 \rightarrow b$

$$(\square \times \square) - (1968 \times \square) + 1967 = 0; \quad \{1967, 1\}$$

(c) UV:  $0 \rightarrow a, -16 \rightarrow b$

$$(\square \times \square) + -16 = 0; \quad \{4, -4\}$$

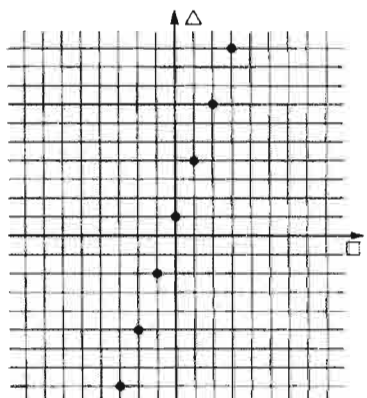
(d) UV:  $5 \rightarrow a, 0 \rightarrow b$

$$(\square \times \square) - (5 \times \square) = 0; \quad \{5, 0\}$$

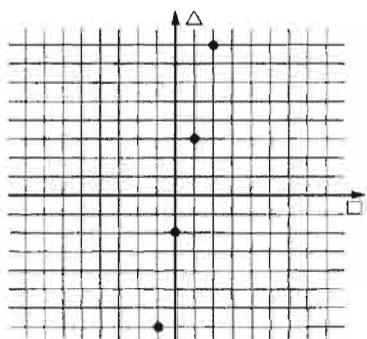
(2) This will depend upon your class.

(3) Descartes would have said that  $a$  and  $b$  stand for constants, whereas  $\square$  stands for "the unknown" or "a variable."

(4) The graph will intersect the vertical axis at  $(0, b)$ .



$$(3 \times \square) + 1 = \triangle$$



$$(5 \times \square) + 1 = \triangle$$

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We can say that if the equation is

$$(a \times \square) + b = \triangle .$$

then the "slope" or "steepness" pattern is: "over one to the right, and up  $a$ ." The number  $b$  also has a geometric significance. Do you know what it is?

(5) In question 4, which would Descartes call variables and which would he call constants?

(6) Geoff says that before you start in on working a problem, you choose definite numbers for the constants. While you are working that one problem, you don't change these constant numbers. But once you finish that one problem, if you want to go on to another problem, you may choose new constants. What do you think?

(7) Can you give some examples of variables and constants?

(5) The  $a$  and  $b$  stand for constants, whereas the  $\square$  and  $\triangle$  stand for variables. Or, as we have said, the  $a$  and  $b$  specify the particular *line* we are talking about, whereas the  $\square$  and  $\triangle$  determine points along that line.

(6) Geoff's description is a very good one (but be forewarned — exceptions do occur!).

(7) In addition to examples occurring earlier, here are some.

(a) In our work on *guessing functions*, whenever we guessed the *form* first and then guessed the actual numbers later, we were working with constants. The forms, as we did write them, and as we should have written them include:

As we did write them

$$\begin{aligned}
(\_)\_ + \frac{(\_)}{x} &= y \\
(\_)\_ + (\_ \times \square) &= \triangle \\
(\_)\_ + (\_)\_ \times (\square \times \square) \\
&= \triangle \\
(\_)\_ + (\_ \times \square) + (\_ \times \square^2) \\
&= \triangle
\end{aligned}$$

As we should have written them

$$\begin{aligned}
a + \frac{b}{x} &= y \\
a + (b \times \square) &= \triangle \\
a + b \times \square^2 &= \triangle \\
a + bx + cx^2 &= y \\
\text{or} \\
a + (b \times \square) \\
+ (c \times \square^2) &= \triangle
\end{aligned}$$

(b) In *Discovery*, Chapter 37, we studied what we called "machines." What was meant was a process in several stages.

Stage 1: We solved various equations, all with the same pattern:

$$\begin{aligned}
\square + 3 &= 5 & \{2\} \\
\square + 21 &= 100 & \{79\} \\
\square + 7 &= -12 & \{-5\} \\
&\vdots \\
&\vdots
\end{aligned}$$

Stage 2: We used variables and constants to write the *general form* for all of the equations

$$\square + a = b.$$

Stage 3: We found the *general solution* to the general form of the problem

$$\square + a = b \quad \{b - a\}.$$

Stage 4: The *general solution* can now be used to solve any *specific problem* by using nothing more than UV. Suppose we wish to solve

$$\square + +31 = -204.$$

We look at the general solution

$$\square + a = b \quad \{b - a\}$$

and use UV

$$\begin{aligned}
\text{UV: } +31 &\rightarrow a \\
-204 &\rightarrow b
\end{aligned}$$

The truth set is  $\{-204 - +31\}$ , which can also be written  $\{-235\}$ . These "general solutions" are also often called "formulas."

## PLATO'S ARISTOCRATS AND TODAY'S DIGITAL COMPUTERS

We continue the work on the general form of a problem and on its general solution, which we began in Chapter 31. (See also *Discovery*, Chapters 37, 38, 41, and 50.)

Plato, of course, wrote about *aristocrats* or *rulers* who determined what was to be done (in the sense of *strategy*), and *administrators* or *auxiliaries* who attempted to carry out the policies of the rulers.

A somewhat parallel situation can be found in two instances in today's technology. In the first instance, we have scientists and creative engineers who attempt to determine *what is to be done*, and then embody this knowledge in handbooks, textbooks, and manuals of various sorts. We then have technicians, assistants, or other engineers who try to operate according to the knowledge that is contained within the handbooks and manuals. The parallel to Plato's society should be clear.

There is a second parallel today, for we have machines—digital computers—that do what they are told, and we have human beings (computer programmers) whose job it is to tell the machines what to do.

In traditional mathematics curricula most students never did learn *why* we use "formulas." In effect, the "formula" or "general solution" is a communication from the *strategist* who made it up to the *technician* who is to use it. We try to stress this distinction in the present chapter. (Actually, of course, the same individual person may function sometimes as strategist and at other times as technician, just as he may be sometimes a pedestrian and at other times a motorist.)



### CHAPTER 32

## Plato's Aristocrats and Today's Digital Computers

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In his book entitled *The Republic*, the ancient Greek philosopher Plato (born 427 B.C.; died 347 B.C.) wrote about the roles of the "aristocrats" and the "nonaristocrats." You can read about this in a short modern essay, entitled *The Teaching of Science as Enquiry* by J. J. Schwab.\*

\*J. J. Schwab, "The Teaching of Science as Enquiry" (included in *The Teaching of Science* by J. J. Schwab and Paul F. Brandwein, Harvard University Press, Cambridge, Mass., 1962).

If you want to read what Plato himself had to say, there are many excellent editions of *The Republic* that are available. One, in paperback, is Plato, *The Republic*, translated by H. D. P. Lee (Penguin Books, Baltimore, Md., 1955).

## ANSWERS AND COMMENTS

To get some idea of what Schwab (and Plato) are talking about, imagine that Mr. Hawkins is a scientist and that he has an assistant who knows

- (a) the basic rules of arithmetic,
- (b) how to use UV (use of variables) correctly;

but outside of this the assistant does not know any algebra.

Now, Mr. Hawkins realizes that they will soon need to solve a large number of problems, all of the same type, like this:

$$3 + \square = 5$$

$$21 + \square = 38$$

$$9 + \square = 11$$

$$6 + \square = 101$$

Consequently, Mr. Hawkins writes on a slip of paper

$$a + \square = b$$

$$\{ b - a \}$$

and gives it to his assistant.

(1) Will the assistant now be able to help solve these problems?

(2) Suppose you want your assistant to help you solve problems like these:

$$3 \times \square = 27$$

$$2 \times \square = 4$$

$$5 \times \square = 21$$

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What could you write on a piece of paper so that your assistant will be able to help you, if he uses only UV and simple arithmetic?

(3) Suppose you want your assistant to help you solve problems like these:

$$(3 \times \square) + 5 = 11$$

$$(2 \times \square) + 15 = 29$$

$$(5 \times \square) + 6 = 17$$

What could you write on a piece of paper so that your assistant will be able to help you, if all he knows is simple arithmetic and how to use UV correctly?

(1) He should be able to solve the problems, since all that is now required is the use of UV, followed by appropriate operations of arithmetic, and we assume the assistant is competent to handle such matters.

(2) You could write:

When you encounter an open sentence of the form

$$a \times \square = b,$$

remember that the truth set is

$$\left\{ \frac{b}{a} \right\}.$$

Caution: This method will *not* work if  $a = 0$ .

(3) You might write:

Whenever you encounter an equation like

$$(a \times \square) + b = c,$$

remember that the truth set is

$$\left\{ \frac{c - b}{a} \right\}.$$

Beware: This method will not work if  $a = 0$ .



(4) Do you know what mathematicians mean when they talk about "the general form of a problem"? What do they mean when they say "the general solution of a problem"?

- (4) I'll leave this to you to try to clarify in discussion with your students. As an example, see question 3. Notice that for problems of the general pattern

$$\begin{aligned}(3 \times \square) + 5 &= 11 \\ (2 \times \square) + 15 &= 29 \\ &\vdots \\ &\vdots\end{aligned}$$

the general form of the problem is

$$(a \times \square) + b = c,$$

and the general solution is

$$\left\{ \frac{c - b}{a} \right\}.$$

(5) Mathematicians call the equation  $3x + 5 = 0$  linear (or "of degree one"), they call the equation  $x^2 - 5x + 6 = 0$  quadratic (or "of degree two"), and they call the equation  $x^3 - 2x^2 + 3x - 5 = 0$  cubic (or "of degree three").

- (5) Using our distinction between constants and variables, we can say that

$\alpha = y$  or  $\alpha = \triangle$  is of degree zero (a constant function),

$\alpha x + b = y$  or  $(a \times \square) + b = \triangle$  is of degree one (linear),

$\alpha x^2 + bx + c = y$  or  $(a \times \square^2) + (b \times \square) + c = \triangle$  is of degree two (quadratic),

$\alpha x^3 + bx^2 + cx + d = y$  or  $(a \times \square^3) + (b \times \square^2) + (c \times \square) + d = \triangle$  is of degree three (cubic),

and so on.

The classification for equations is similar to the classification for functions:

$\alpha x + b = 0$  or  $(a \times \square) + b = 0$  is a first-degree equation (linear),

$\alpha x^2 + bx + c = 0$  or  $(a \times \square^2) + (b \times \square) + c = 0$  is a second-degree equation (quadratic),

$\alpha x^3 + bx^2 + cx + d = 0$  or  $(a \times \square^3) + (b \times \square^2) + (c \times \square) + d = 0$  is a third-degree equation (cubic),

and so on.

With this system of classification, not every equation (nor every function) has a degree. For example, none of the following do.

$$\begin{aligned}|3x + 2| &= 5 \\ \frac{7x^2 + 3x + 2}{2x^3 - 1} &= y\end{aligned}$$

$$|3x + 2| = y$$

$$y = \frac{1}{x} + \frac{2}{x^2}$$

$$y = \sin x$$

We could easily extend our system of classification to include a few more equations and functions, but we shall not bother to do so at this time.

What do you suppose they would call each of the following?

- (a)  $x^2 - 13x + 22 = 0$
- (b)  $x^3 + 2x^2 = 16$
- (c)  $x - 2 = 0$
- (d)  $3 + x = 7$
- (e)  $x^2 = 4$
- (f)  $x^4 + x^2 - x = 1$
- (g)  $x^7 + x^6 - x^3 + x + 9 = 0$
- (h)  $x^3 - x^2 + x - 1 = 0$

(6) How could you write the general form for a linear equation?

(7) Could you write the general solution for the general linear equation?

(8) How could you write the general form for a quadratic equation?

(9) How could you write the general solution for the general quadratic equation?

(10) How could you write the general cubic equation? the general fourth-degree equation? the general fifth-degree equation?

- (a) Quadratic (degree two)
- (b) Cubic (degree three)
- (c) Linear (degree one)
- (d) Linear (degree one)
- (e) Quadratic (degree two)
- (f) Quartic (degree four)
- (g) Degree seven
- (h) Cubic (degree three)

(6)  $ax + b = 0$  or  $(a \times \square) + b = 0$

(The general linear function would be  $ax + b = y$ .)

(7)  $ax + b = 0, \left\{ \frac{b}{a} \right\}$  Note:  $a$  must not equal zero.

(8) There are several possibilities. The most common is

$$ax^2 + bx + c = 0,$$

where  $a \neq 0$ , but we could instead write

$$x^2 + ax + b = 0.$$

In the present book, we shall find it convenient to write

$$x^2 - ax + b = w.$$

You may need to think—and to write—for a few moments to see why these three different forms really say the same thing.

(9) We hope that, at this stage, your students will not be able to answer this question. (This question is inserted at this point in order to help the student see where he stands, by locating the "boundary" or "frontier" of his present knowledge. He *can* solve the general linear equation; he (presumably) *cannot* solve the general quadratic equation.)

(10) Again, there are many possibilities. Here is one:

General cubic  $x^3 + ax^2 + bx + c = 0$

General fourth-degree equation  $x^4 + ax^3 + bx^2 + cx + d = 0$

General fifth-degree equation  $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$

(11) How could you write the general solution for each type of equation in question 10?

(11) Presumably at this stage your students can solve none of these.

Some people may wonder why we include questions that students can't answer. We feel very strongly about this: *if you don't know the distinction between what you do know and what you don't know, then all of your knowledge is suspect.*

When, for example, you go to a physician, you are trusting him to refer you to a specialist whenever your illness lies outside the range of his own knowledge. We should all try to advance the frontiers of our knowledge as far as possible, *but we should know where those frontiers lie.*

## HINTS ON HOW TO SOLVE PROBLEMS

When they are confronted by a problem, one of the most common complaints of students is, "I don't know where to start!" One common answer is *to show* the student where to start.

In our view, this answer is usually unsatisfactory. It's too *specific*. It may get the student started on *that* problem – but how long will it be before he encounters some *new* problem where, once more, he says, "I don't know where to start"?

Actually, there is a better approach. This is the approach which has been described in detail by Professor George Polya, and by others. It consists of replacing one "hard" question by a *sequence of easier questions*. This method is of the greatest importance. If we do not show it to our students, we leave them seriously handicapped for all their future work in solving new kinds of problems.



### CHAPTER 33

## Hints on How To Solve Problems

[page 105]

One of the very important mathematicians of this century is George Polya, of Stanford University. Professor Polya has succeeded in solving many problems that no one else had ever been able to solve. He has reflected on this experience, and written a number of suggestions on how to go about solving difficult or puzzling problems.\*

The kind of suggestions Professor Polya has made go somewhat like this:

- (a) If the problem is too hard, can you see which part of the problem is making it hard?
- (b) Is there any way to eliminate (or ignore) the "hard part"? Could you solve the easy part of the problem?
- (c) Could you find a different problem that might be easier? (After you've solved this easier problem you may have learned enough so that then you can solve

\*George Polya, *How to Solve It* (Doubleday Anchor Books, New York, 1957).  
George Polya, *Mathematics and Plausible Reasoning*, Two volumes (Princeton University Press, Princeton, N.J., 1954).

George Polya, *Mathematical Discovery. On Understanding, Learning, and Teaching Problem Solving*, Two volumes (John Wiley and Sons, New York, 1965).

## ANSWERS AND COMMENTS

the hard problem. Obviously, you want the easy problem to be at least a little bit similar to the original hard problem.)

(d) Can you change this new problem around, so that it will turn into some kind of problem that you already know about? This is often called "reducing it to a problem that has already been solved."

(e) If you succeed in solving one kind of problem, you may want to ask yourself if what you have just done might let you go forward and solve some other harder (or more general) problems.

In the next chapter, we want to work on a famous mathematical problem, namely, the task of finding the general solution of the general quadratic equation. Professor Polya's suggestions can help us.

First, however, it may be wise to practice using some of his methods.

(1) Solving the general quadratic equation is a fairly hard problem. Perhaps we should start with some easier ones. Are there any quadratic equations that you already know how to solve? Can you make up some "easy" quadratic equations that you can easily solve right now?

- (1) This first question is a good example of what we mean. Suppose someone asks you to solve the general quadratic equation. That is a hard problem. Many—probably most—students would have the feeling that they didn't know where to begin.

But let's try to replace this hard question by several easy questions. Do we know what is meant by a "quadratic equation"? (If not, reread the answer to question 5, Chapter 32.) Can you write down a quadratic equation or two, so we can get a look at one? Here are some:

$$\begin{aligned}x^2 - 5x + 6 &= 0 \\23x^2 + 1961x + 1066 &= 1,000,037 \\x^2 - 4 &= 0 \\x^2 &= 9 \\x^2 - x &= 0 \\x^2 - 2x + 1 &= 0 \\x^2 - 4x + 4 &= 0 \\x^2 - 20x + 96 &= 0\end{aligned}$$

Now that we've had a look at a few quadratic equations, let's see if we can write down a few *really easy* ones that we'll be able to solve without difficulty. Here are some:

$$\begin{aligned}x^2 - 5x + 6 &= 0 \\x^2 - 15x + 26 &= 0 \\x^2 - 7x + 10 &= 0\end{aligned}$$

These are all easy; in fact, we made them up by thinking of the truth set first, and then writing down the equation. For example (see Chapter 10), if we want the truth set to be  $\{2, 3\}$  we take the equation

$$x^2 - \text{---}x + \text{---} = 0$$

$\uparrow$                        $\uparrow$   
 Some                      Some  
 number                      number

and find the missing numbers by saying  $2 + 3 = 5$  and  $2 \times 3 = 6$ , and so we have  $x^2 - 5x + 6 = 0$ .

Are there any other *easy* kinds of quadratic equations? Yes; if the equation is factored, as in

$$(x - 5) \cdot (x - 27) = 0,$$

the truth set can be found at once; in this example, it is  $\{5, 27\}$ . Furthermore, if in the equation

$$x^2 - ax + b = w$$

we use

$$\begin{aligned} UV: 0 &\rightarrow a \\ 0 &\rightarrow w, \end{aligned}$$

we have

$$x^2 + b = 0.$$

Now, if  $b$  is negative, and a perfect square, as in

$$x^2 - 16 = 0,$$

the truth set can be written down immediately; in this example, it is  $\{+4, -4\}$ .

We now have three leads that we can follow: the first involves the coefficient rules

$$\begin{aligned} x^2 - ax + b &= 0 \\ &\{r_1, r_2\} \\ r_1 + r_2 &= a \\ r_1 \cdot r_2 &= b; \end{aligned}$$

the second involves *factoring* [as in

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x - 2) \cdot (x - 3) &= 0 \\ &\{2, 3\}; \end{aligned}$$

and the third involves *perfect squares*.

We could follow any one of these three paths, but the one we have chosen to follow is the third: looking for equations that involve perfect squares.

(2) Solving the general cubic equation is also a hard problem. Are there any cubic equations that you can solve right now?

(2) Well, if we know how to use the coefficient rules in this case, we can think of the answer first, and then make up a problem to go with it. Now, in this book (and in *Discovery*) we have never studied the coefficient rules for cubic equations. However, such rules *do* exist; in fact, we can figure them out.

If the truth set is

$$\{r_1, r_2, r_3\},$$

then the equation is

$$(x - r_1) \cdot (x - r_2) \cdot (x - r_3) = 0.$$

If we "multiply this out," we get

$$\begin{aligned} &(x^2 - (r_1 + r_2)x + r_1r_2) \cdot (x - r_3) = 0 \\ &x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x \\ &\quad - r_1r_2r_3 = 0. \end{aligned}$$

Consequently, if we want the truth set to be  $\{2, 3, 4\}$ , then the equation should be

$$x^3 - 9x^2 + 26x - 24 = 0.$$

Well, that method may (or may not) look very promising. Let's see if we can find some others.

If the equation is *factored*, as in

$$(x - 5) \cdot (x - 1) \cdot (x - 10) = 0,$$

then we can write down the truth set immediately; in the present example, it is  $\{5, 1, 10\}$ .

If the equation involves a perfect cube, then we can write down *one* element of the truth set immediately; for example,  $x^3 = 8$ ,  $\{2, r_2, r_3\}$ , where we know the root 2, but we do not immediately see what  $r_2$  or  $r_3$  should be. (In fact, they are more complicated.)

Everything considered, *cubic equations* are harder than *quadratic equations*, so let's not try to pursue them too far just yet.

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(3) Let's try reducing a new problem to an old one that you already know how to solve. Suppose that your assistant has a piece of paper that tells him how to solve equations like this:

$$\square + 3 = 10$$

$$\square + 5 = 10$$

$$\square + 1 = 10$$

Here, in fact, is what the paper says:

$$\begin{array}{c} \square + a = 10 \\ \{10 - a\} \end{array}$$

Now, one day your assistant is confronted by the problem

$$\square + 18 = 25.$$

He can't do this by merely using UV and the formula on his piece of paper. Why not? What can he do?

(4) Can you extend your solution to harder or more general problems? You already know how to solve the equations

$$x^2 = 9$$

$$x^2 = 16$$

$$x^2 = 121$$

and so on.

(3) If he knows about *transform operations*, he can subtract 15 from both sides of his equation, getting

$$\square + 3 = 10.$$

Now, this new equation *is* of the type that he can solve by looking at his piece of paper and using UV. Using UV:  $3 \rightarrow a$ , the truth set is  $\{10 - 3\}$ , or  $\{7\}$ .

In Polya's language, the assistant has reduced this new problem to an old problem that he already knew how to solve.

If your class is confused by this problem, and if you want to give them a hint, you might say, "Suppose the assistant knows about transform operations. Would that help him?"

(4) **Yes.** There are several ways to write this, or to think about it. One is to say that for the equation  $(x - 2)^2 = 49$ , the number for the term  $(x - 2)$  must be either '7 or '7. Therefore, either  $x - 2 = '7$  or  $x - 2 = ^-7$ . If  $x - 2 = '7$ , then  $x = '9$ . If  $x - 2 = ^-7$ , then  $x = ^-5$ . Hence, for the equation  $(x - 2)^2 = 49$ , the truth set is  $\{'9, ^-5\}$ .

Could you use this to help you solve

$$(x - 2)^2 = 49?$$

(5) Again, try reducing a new problem to an old one that you already know how to solve. If you can solve the equation

$$(x - 3)^2 = 121,$$

can you use that to help you solve

$$x^2 - 6x + 9 = 144?$$

(6) Can you solve the equation

$$x^2 - 6x + 7 = 79?$$

Notice that we could generalize this still further. The number here

$$(x - \quad) = 49$$

didn't *have* to be 2. It could have been any number:

$$(x - a)^2 = 49.$$

Then either  $x - a = +7$  or  $x - a = -7$ . Hence, either  $x = a + 7$ , or  $x = a - 7$  (which we might prefer to write as  $x = a + 7$  or  $x = a - 7$ ). Hence, for the equation  $(x - a)^2 = 49$ , the truth set is  $\{a + 7, a - 7\}$ .

Question: Does the 49 *have* to be 49? Could you generalize this method still further?

(5) Yes, you can turn this unfamiliar new problem into a familiar old one, by using the identity

$$x^2 - 6x + 9 = (x - 3)^2.$$

Here is how it goes:

(i)  $x^2 - 6x + 9 = 144$

(ii)  $x^2 - 6x + 9 = (x - 3)^2$

(iii)  $(x - 3)^2 = 144$  PN, from line (i), using line (ii).

Now, the problem

$$(x - 3)^2 = 144$$

is a familiar kind of problem that we can easily solve (see question 4).

(6) This is a very good illustration of Professor Polya's method of breaking one big question down into several little ones. Using our method of looking for perfect squares, we know we *could* solve a quadratic equation like  $x^2 = 144$ , getting  $\{+12, -12\}$ .

We could extend the method, and solve  $(x - 2)^2 = 144$ , getting  $\{+12 + 2, -12 + 2\}$ , or  $\{+14, -10\}$ .

We could even solve  $x^2 - 4x + 4 = 144$ , since  $x^2 - 4x + 4 = (x - 2)^2$ .

Now, how about  $x^2 - 6x + 7 = 79$ ? Can we use an identity of the type

$$x^2 - 6x + 7 = (x - \quad)^2?$$

Let's look at a few identities of this type to see how they work:

$$x^2 - 4x + 4 = (x - 2)^2$$

$$x^2 - 8x + 16 = (x - 4)^2$$

$$x^2 - 14x + 49 = (x - 7)^2$$

⋮



Evidently, the number that goes here

$$x^2 - \text{---} x + \text{---} = (x - \text{---})^2$$

↑

must be one-half of the number that goes here

$$x^2 - \text{---} x + \text{---} = (x - \text{---})^2$$

↑

Hence, in our present problem, the missing number must be 3:

$$x^2 - 6x + 7 = (x - 3)^2$$

↑

Now, if we look back at the patterns for this kind of identity, we see that the number that goes here

$$x^2 - \text{---} x + \text{---} = (x - \text{---})^2$$

↑

must be *the square* of the number that goes here

$$x^2 - \text{---} x + \text{---} = (x - \text{---})^2$$

↑

Hence, in our present problem, this number

$$x^2 - 6x + 7 = (x - 3)^2$$

↑

must be the square of this number

$$x^2 - 6x + 7 = (x - 3)^2$$

↑

Unfortunately, it isn't:  $7 \neq 3^2$ .

Indeed, we can now answer three Polya-type questions:

(i) What's *difficult* about this problem? Answer: It has a 7 here:

$$x^2 - 6x + 7 = 79$$

↑

(ii) What number would you *like* to see in place of the 7? Answer: 9.

(iii) Is there any legal way to get a 9 there? Answer: Yes. Add 2 to each side of the original quadratic equation

$$x^2 - 6x + 7 = 79,$$

to get

$$x^2 - 6x + 9 = 81.$$

(iv) Now we have turned this *unfamiliar new* problem into a *familiar old* kind that we already know how to solve! Here we go:

(i)  $x^2 - 6x + 9 = 81$

(ii)  $x^2 - 6x + 9 = (x - 3)^2$

(iii)  $(x - 3)^2 = 81$  PN, from line (i), using line (ii).

This last equation means that this number

$$\left( \quad \right)^2 = 81$$

↑

must be  $+9$  or it must be  $-9$ . Hence, either  $x - 3 = +9$  or  $x - 3 = -9$ . In the first case,  $x = +12$ ; in the second case,  $x = -6$ . Hence, the truth set for  $x^2 - 6x + 9 = 81$  must be  $\{+12, -6\}$ .

Now, we obtained the equation  $x^2 - 6x + 9 = 81$  by adding 2 to each side of the equation  $x^2 - 6x + 7 = 79$ . Since adding 2 to each side is a *transform operation*, the equation  $x^2 - 6x + 7 = 79$  must have the same truth set as the equation  $x^2 - 6x + 9 = 81$ . Consequently,  $x^2 - 6x + 7 = 79$  must have the truth set  $\{+12, -6\}$ .

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(7) Your assistant has a piece of paper which says:

For the equation  
 $x^2 - ax + b = w,$

the truth set is  
 $\left\{ \frac{a}{2} + \sqrt{w}, \frac{a}{2} - \sqrt{w} \right\}.$

\* Note! This method  
 will work if  
 $\left( \frac{a}{2} \right)^2 = b.$

This method will not  
 work if  
 $\left( \frac{a}{2} \right)^2 \neq b.$

BEWARE! 

(7) He should observe that  $\frac{a}{2} = 4$  and that  $4^2 = 16 \neq 10$ . Hence, he cannot take the rule on the paper and use it for the equation

$$x^2 - 8x + 10 = 19.$$

Instead of a 10 here

$$x^2 - 8x + 10 = 19,$$

↑

he wants a 16. Consequently, he should add 6 to each side of the equation and then proceed according to the instructions on his paper. (See question 6.)

Now, your assistant runs into this problem:

$$x^2 - 8x + 10 = 19$$

What should he do?

## ALL THE QUADRATIC EQUATIONS IN THE WORLD



### CHAPTER 34

## All the Quadratic Equations in the World

[page 108]

In this chapter, we want to solve the general quadratic equation. We'll try to proceed by small steps, and make use of Professor Polya's suggestions.

(1) How can you write the general quadratic equation?

(2) Can you think of any quadratic equations that are so easy that you can solve them just by looking at them?

## ANSWERS AND COMMENTS

(1) There are many possibilities. We prefer to use

$$x^2 - Ax + B = W,$$

since this will be convenient for the work of the rest of this chapter.

(2) As we saw in the preceding chapter, we have three possible lines of attack:

Using coefficient rules

$$x^2 - 5x + 6 = 0$$

$$2 + 3 = 5$$

$$2 \times 3 = 6$$

Factoring

$$(x - 2) \cdot (x - 3) = 0$$

Looking for perfect squares

$$x^2 = 16$$

In this book we shall follow the third line of attack (seeking perfect squares, a method known to ancient mathematicians as early as 2000 B.C.\*). If any of your more capable students wish, they can write a chapter parallel to this one, in which they use the method of factoring as their basic line of attack.†

\*See Eves (151).

†This is not really very different from our present method. In order to be sure that

$$x^2 - Ax + B = 0$$

can be factored, you need to select a form that you know will factor. The simplest choice is probably a difference of squares:

$$x^2 - R^2 = (x - R) \cdot (x + R).$$

This means writing your equation of the form

$$(x - \alpha)^2 - \beta^2 = 0.$$

The rest is reasonably straightforward.

Since we wish to follow the lead of seeking perfect squares, we shall look carefully at all three easy types just mentioned, but it is the type

$$x^2 = 16$$

which we shall especially pursue.

Having solved

$$x^2 = 16,$$

we shall now seek to *generalize* this (see problem 4 of Chapter 33).

- (3) Can you find the truth set for the open sentence

$$x^2 = 16?$$

- (4) Can you find the truth set for the open sentence

$$(x - 1)^2 = 49?$$

- (5) Can you find the truth set for the open sentence

$$(x - 3)^2 = 144?$$

- (6) Can you find the truth set for the open sentence

$$(x - 2)^2 = 81?$$

- (7) Paul wrote this on a piece of paper:

For the open sentence

$$(x - p)^2 = 144,$$

the truth set is

$$\{p + 12, p - 12\}.$$

Then Paul gave the paper to his assistant. Suppose his assistant needs to solve the equation

$$(x - 2)^2 = 144.$$

Can he do it?

- (3)  $\{+4, -4\}$

- (4) We proceed as in question 4, Chapter 33. Since  $(x - 1)^2 = 49$ , the number inside the parentheses must be either  $+7$  or  $-7$ . Hence, either  $x - 1 = +7$  or  $x - 1 = -7$ . In the first case, we get  $x - 1 = +7$ ,  $x = +8$ ; in the second case, we get  $x - 1 = -7$ ,  $x = -6$ . Hence, the equation  $(x - 1)^2 = 49$  has the truth set  $\{+8, -6\}$ .

You may prefer to have students think about this problem intuitively, without going into any verbal explanations such as this.

- (5) Since  $(x - 3)^2 = 144$ , the number inside the parentheses must be either  $+12$  or  $-12$ . Hence, either  $x - 3 = +12$  or  $x - 3 = -12$ . We can solve the two linear equations above by adding 3 to each side, getting  $x = +15$  or  $x = -9$ . Hence, the equation  $(x - 3)^2 = 144$  has the truth set  $\{+15, -9\}$ .

- (6) This is precisely similar to questions 4 and 5. You can discuss it in exactly the same way. However, for the sake of variety, we show a method which mathematicians often use, called "changing variables."

Since "the number inside the parentheses" is  $x - 2$ , we introduce a new variable  $t$  by letting  $t = x - 2$ . Then  $t^2 = 81$ . Hence, either  $t = +9$  or  $t = -9$ . Now, using the definition of  $t$  ( $t = x - 2$ ) we get either  $x - 2 = +9$  or  $x - 2 = -9$ . Solving these last two linear equations, we get either  $x = +11$  or  $x = -7$ . Hence, the equation  $(x - 2)^2 = 81$  has the truth set  $\{+11, -7\}$ .

- (7) Yes. He uses UV:  $2 \rightarrow p$  to get the statement that the open sentence  $(x - 2)^2 = 144$  has the truth set  $\{2 + 12, 2 - 12\}$ , which can be written as  $\{+14, -10\}$ .

(8) Jerry says that Paul was foolish to write "144" on the paper. What do you think?

(8) Jerry has a point. The number here

$$(x - p)^2 = 144$$

doesn't really have to be 144. Any number  $k$  will do, provided we can find the square root of  $k$ . If, instead of writing 144 here

$$(x - p)^2 = 144,$$

Paul had written

$$(x - p)^2 = k,$$

then he would have solved a *more general problem* — he would have given his assistant a more powerful procedure. The difference, of course, is that Paul's assistant, as matters stand, cannot solve

$$\begin{aligned} (x - 2)^2 &= 49 \\ (x - 3)^2 &= 36 \\ (x - 5)^2 &= 121 \\ &\vdots \end{aligned}$$

If Paul had followed Jerry's suggestion, the assistant *would* be able to solve all of these.

(9) What do you suppose Jerry wrote on the paper that he gave to his assistant?

(9) Presumably, Jerry wrote:

If you encounter the equation

$$(x - p)^2 = k$$

(where  $p$  and  $k$  are given numbers), remember that the truth set is

$$\{p + \sqrt{k}, p - \sqrt{k}\}.$$

Modern mathematicians use the symbol  $\sqrt{k}$  to refer to the *non-negative square root* of  $k$ . Thus,  $\sqrt{144} = +12$ . Do not use  $\sqrt{144}$  to stand for  $-12$ ; this would be contrary to the best present practice.

Jerry's method is not perfect; it works very nicely, *provided* we are able to find the square root,  $\sqrt{k}$ . If, however, we *cannot* find this square root, the method fails.

When can we find square roots, and when are we unable to? This, of course, depends upon the sequencing of the mathematics curriculum in your school. In our own practice, *at this stage in a student's career*, he can find the square roots of perfect squares, such as

$$\begin{aligned} \sqrt{121} &= 11 \\ \sqrt{4} &= 2 \\ \sqrt{9} &= 3 \end{aligned}$$

and so on, but these are the *only* square roots he can find.

In fact, he can prove (more or less, anyhow) that, *among the numbers that he knows*, there is no number whose square is 2, and there is no number whose square is  $-4$ .

The introduction of matrices will put matters in a different light — but we haven't reached this point yet.

(10) Suppose the next problem was

$$(x - 5)^2 = 81.$$

Can Paul's assistant solve this? Can Jerry's assistant solve it?

[page 109]

(11) For the problem

$$(x - 1)^2 = 144,$$

Paul's assistant looked at his paper

For the open sentence  
 $(x - p)^2 = 144,$   
 the truth set is  
 $\{p + 12, p - 12\}.$

and wrote:

UV:  $1 \rightarrow p$   
 $\{1 + 12, 1 - 12\}$   
 $\{13, -11\}$

Did the assistant do the right thing?

(12) Pretend you are Paul's assistant. (That means you know about UV and simple arithmetic, but you don't know anything about equations unless you can read it on a slip of paper!) How would you solve

$$(x - 11)^2 = 144?$$

(13) Stop pretending you are Paul's assistant. Now you are a very clever scientist. Can you find the truth set for the open sentence

$$(x - h)^2 = k?$$

(Remember, in Descartes' words  $h$  and  $k$  are "constants." That means that somebody else will put numbers in for  $h$  and  $k$ , before they give you the problem.)

What will you write on the slip of paper you give to your assistant?

(10) Presumably, Paul's assistant cannot solve this, since it involves an 81 at a spot in the equation where Paul's assistant can cope only with 144.

Jerry's assistant, on the other hand, can solve

$$(x - 5)^2 = 81,$$

by using UV as follows:

$$\begin{array}{l} \text{UV: } 5 \rightarrow p \\ \quad \quad 81 \rightarrow k \end{array}$$

in his equation  $(x - p)^2 = k.$

Then  $\sqrt{k} = \sqrt{81} = 9$  and the truth set is  $\{5 + 9, 5 - 9\}$ , which can be written  $\{14, 4\}.$

(11) Yes

(12) You would use UV, UV:  $11 \rightarrow p$ , in the equation  $(x - p)^2 = 144$  so that the truth set would be  $\{11 + 12, 11 - 12\}$ , which can be written  $\{23, -1\}.$

(13) Give your assistant a slip of paper with the following written on it:

If you want to solve the equation

$$(x - h)^2 = k,$$

where you are told definite numbers for  $h$  and  $k$ , and you are trying to find values for  $x$ , then the truth set is

$$\{h + \sqrt{k}, h - \sqrt{k}\}.$$

That means that if you put the number  $h + \sqrt{k}$  in for  $x$ , you will get a true statement, and similarly for  $h - \sqrt{k}.$

If you have trouble seeing why we wrote this, you can think about the equation

$$(x - h)^2 = k$$

the same way that we did with problems 4, 5, and 6, earlier in this chapter.

(14) Pretend you are Jerry's assistant. How will you solve each of the following equations? (Whenever you use UV, write it down as UV:  $3 \rightarrow h$ , substituting for 3 whatever number you do use, and so on.)

(a)  $(x - 1)^2 = 9$

(b)  $(x - 4)^2 = 169$

(c)  $(x - 15)^2 = 225$

(d)  $(x - 10)^2 = 9$

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Before we can go any further, we'll need to use some identities. It might be a good idea to practice a few identities right now.

(15) What do we mean by an identity?

(16) Is this an identity?

$$(\square - 3) \times (\square - 3) = (\square \times \square) - 9$$

(14) Jerry's assistant can use UV, as follows:

(a)  $(x - 1)^2 = 9$   
 UV:  $1 \rightarrow p$   
 $9 \rightarrow k$   
 $\sqrt{k} = \sqrt{9} = +3$   
 $\{1 + 3, 1 - 3\}$ , or  $\{4, -2\}$

(b)  $(x - 4)^2 = 169$   
 UV:  $4 \rightarrow p$   
 $169 \rightarrow k$   
 $\sqrt{k} = \sqrt{169} = +13$   
 $\{4 + 13, 4 - 13\}$ , or  $\{17, -9\}$

(c)  $(x - 15)^2 = 225$   
 UV:  $15 \rightarrow p$   
 $225 \rightarrow k$   
 $\sqrt{k} = \sqrt{225} = 15$   
 $\{15 + 15, 15 - 15\}$ , or  $\{30, 0\}$

(d)  $(x - 10)^2 = 9$   
 UV:  $10 \rightarrow p$   
 $9 \rightarrow k$   
 $\sqrt{k} = \sqrt{9} = +3$   
 $\{10 + 3, 10 - 3\}$ , or  $\{13, 7\}$

These solutions make use of the "piece of paper" which Jerry gave to his assistant. See question 9, page 303.

(15) Here is one way to say what an identity is:

An identity is an open sentence that becomes *true* whenever you make a legal numerical replacement for the variables.

The students often say this more colloquially as:

**Any number works.**

To say this *very* precisely is difficult, and usually unnecessary. Sufficient unto the day is the rigor thereof. One can always make things *more* precise as the occasion requires.

(16) No. For example, try

UV:  $4 \rightarrow \square$ ,

to get

$$\begin{aligned} (\boxed{4} - 3) \times (\boxed{4} - 3) &= (\boxed{4} \times \boxed{4}) - 9 \\ (4 - 3) \times (4 - 3) &= (4 \times 4) - 9 \\ 1 \times 1 &= 16 - 9 \\ 1 &= 7 \quad \text{False} \end{aligned}$$

Consequently, the open sentence

$$(\square - 3) \times (\square - 3) = (\square \times \square) - 9$$

is *not* an identity (if it were, then putting 4 into the  $\square$  would have yielded a *true* statement).

(17) Is this an identity?

$$(R + S)^2 = R^2 + 2RS + S^2$$

(17) Yes. You can prove this — which takes a bit of writing — by making a derivation. We can sketch out the derivation here in brief form:

- (a)  $(R + S)^2 = (R + S) \times (R + S)$  Definition of the exponent <sup>2</sup>
- (b)  $(R + S)^2 = [(R + S) \times R] + [(R + S) \times S]$  DL
- (c)  $(R + S)^2 = [R \times (R + S)] + [S \times (R + S)]$  CLM<sup>2</sup>
- (d)  $(R + S)^2 = [R^2 + RS] + [SR + S^2]$  DL<sup>2</sup>
- (e)  $(R + S)^2 = \{[R^2 + RS] + SR\} + S^2$  ALA
- (f)  $(R + S)^2 = \{R^2 + [RS + SR]\} + S^2$  ALA
- (g)  $(R + S)^2 = \{R^2 + [RS + RS]\} + S^2$  CLM
- (h)  $(R + S)^2 = \{R^2 + [RS \cdot 1 + RS \cdot 1]\} + S^2$  L1<sup>2</sup>
- (i)  $(R + S)^2 = \{R^2 + (RS \cdot [1 + 1])\} + S^2$  DL
- (j)  $(R + S)^2 = \{R^2 + RS \cdot 2\} + S^2$  Def. Num.
- (k)  $(R + S)^2 = \{R^2 + 2RS\} + S^2$  CLM
- (l)  $(R + S)^2 = R^2 + 2RS + S^2$

The elimination of the braces is possible because of the agreement that  $R^2 + 2RS + S^2$  means carry out the multiplications, then add  $R^2 + 2RS$ , and to this result add  $S^2$ .

Q.E.D.

(18) This is the beginning of a triangular array of numbers that mathematicians call "Pascal's triangle":

$$\begin{array}{c}
 1 \\
 1 \ 1 \\
 1 \ 2 \ 1 \\
 1 \ 3 \ 3 \ 1 \\
 1 \ 4 \ 6 \ 4 \ 1
 \end{array}$$

Can you fill in the next line across? Can you fill in the line after that? and the next?

(18) This triangular array of numbers is of considerable importance in various parts of mathematics. It has been named after the great French mathematician Blaise Pascal (1623-1662), who studied it carefully. Actually, Pascal was not the first mathematician to study this array; according to Eves [Eves (151), pp. 257-261], the earliest known reference is in the work of the Chinese algebraist Chu Shi-kie, in 1303. Here are some additional lines:

$$\begin{array}{c}
 1 \\
 1 \ 1 \\
 1 \ 2 \ 1 \\
 1 \ 3 \ 3 \ 1 \\
 1 \ 4 \ 6 \ 4 \ 1 \\
 1 \ 5 \ 10 \ 10 \ 5 \ 1 \\
 1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1 \\
 1 \ 7 \ 21 \ 35 \ 35 \ 21 \ 7 \ 1 \\
 1 \ 8 \ 28 \ 56 \ 70 \ 56 \ 28 \ 8 \ 1 \\
 1 \ 9 \ 36 \ 84 \ 126 \ 126 \ 84 \ 36 \ 9 \ 1 \\
 \vdots
 \end{array}$$

Have you discovered the pattern that lets you fill in the next line?



(19) Can you see how Pascal's triangle can help you remember these identities?

$$(R + S)^0 = R^0 = 1$$

$$(R + S)^1 = R^1 + S^1$$

$$(R + S)^2 = R^2 + 2RS + S^2$$

$$(R + S)^3 = R^3 + 3R^2S + 3RS^2 + S^3$$

$$(R + S)^4 = R^4 + 4R^3S + 6R^2S^2 + 4RS^3 + S^4$$

(20) Can you write down the missing part for each of these identities?

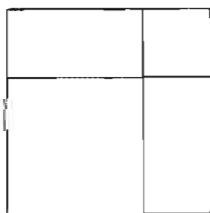
$$(a) \quad (R + S)^3 = R^3 + 5R^2S + 10R^2S^2 + \underline{\hspace{2cm}} + 5RS^4 + S^5$$

$$(b) \quad (R + S)^4 = R^4 + \underline{\hspace{2cm}} + 15R^4S^2 + 20R^3S^3 + \underline{\hspace{2cm}} + 6RS^5 + \underline{\hspace{2cm}}$$

$$(c) \quad (R + S)^7 = R^7 + \underline{\hspace{4cm}}$$

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(21) Which identity in question 19 does this picture suggest?



(19) This is, of course, purely a mnemonic device; it is *not* a proof. Here is how it works:

Suppose we want to expand  $(R + S)^3$ . We know that the first term will be  $R^3$ :

$$(R + S)^3 = R^3 + \underline{\hspace{2cm}}$$

Now, we fill in all the  $R$  and  $S$  terms, omitting (for the moment) their coefficients. There is a pattern to the  $R$  and  $S$  terms, which goes like this:

(a) The exponents of  $R$  decrease as 3, 2, 1, 0:

$$(R + S)^3 = R^3 + \underline{\hspace{1cm}} R^2 \underline{\hspace{1cm}} + \underline{\hspace{1cm}} R \underline{\hspace{1cm}} + \underline{\hspace{1cm}}$$

(b) The exponents of  $S$  increase as 0, 1, 2, 3:

$$(R + S)^3 = R^3 + \underline{\hspace{1cm}} R^2S + \underline{\hspace{1cm}} RS^2 + \underline{\hspace{1cm}} S^3$$

(c) As a check, the last term should be  $S^3$ .

(d) As a further check, the sum of the exponents in any term should be 3:

$R^3$	$3 + 0 = 3$	$(R^3 = R^3S^0)$
$R^2S$	$2 + 1 = 3$	$(R^2S = R^2S^1)$
$RS^2$	$1 + 2 = 3$	$(RS^2 = R^1S^2)$
$S^3$	$0 + 3 = 3$	$(S^3 = R^0S^3)$

We now turn to the task of filling in the coefficients

$$(R + S)^3 = \underline{\hspace{1cm}} R^3 + \underline{\hspace{1cm}} R^2S + \underline{\hspace{1cm}} RS^2 + \underline{\hspace{1cm}} S^3$$

which are given by the line

$$1 \quad 3 \quad 3 \quad 1$$

in Pascal's triangle:

$$(R + S)^3 = 1 \cdot R^3 + 3R^2S + 3RS^2 + 1 \cdot S^3 \\ = R^3 + 3R^2S + 3RS^2 + S^3.$$

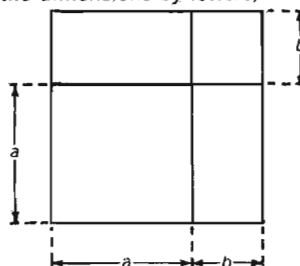
(20) (a)  $10R^2S^3$

(b)  $6R^3S$ ,  $15R^2S^2$ ,  $S^6$

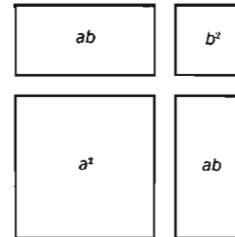
(c)  $7R^6S + 21R^5S^2 + 35R^4S^3 + 35R^3S^4 + 21R^2S^5 + 7RS^6 + S^7$

(21)  $(R + S)^2 = R^2 + 2RS + S^2$

If we indicate the dimensions by letters,



then we see that the largest square is  $a + b$  on a side; hence its area is  $(a + b)^2$ . But this is equal to the sum of the areas of the two smaller squares, plus the two rectangles. These areas are as shown in the following figure.



Thus their sum is  $a^2 + 2ab + b^2$ .

However, the picture seems to show all these little pieces fitting together "just right" to make the big square; if this is correct, their areas must be equal:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

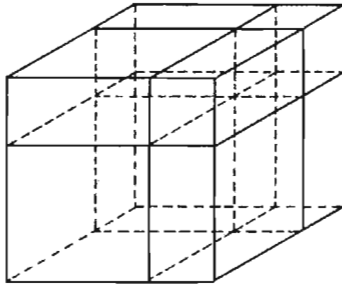
This obviously suggests

$$(R + S)^2 = R^2 + 2RS + S^2;$$

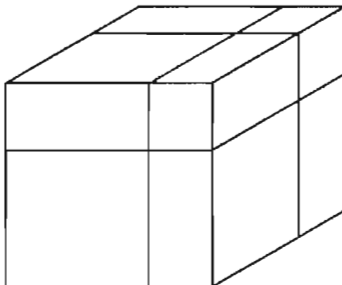
we can get either from the other by using UV.

(22) Here you get your choice of two versions of this picture. Both versions show a cube sliced up into pieces, some of which are themselves cubes.

(a) The cluttered one



(b) The uncluttered one



Whichever picture you choose, they are both supposed to represent the same big block of wood, which has been sliced into smaller pieces.

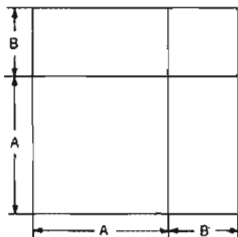
How many small pieces are there?

(22) **There are 8 small pieces that, together, make up the large cube. The large cube itself has the volume  $(A + B)^3$ . The medium-sized cube in the lower left front has the volume  $A^3$ . The small cube in the upper right rear has the volume  $B^3$ . The three "flat" pieces have the volume  $A^2B$ . The three "brick-shaped" pieces have the volume  $AB^2$ .**

Now the picture seems to show all of these smaller pieces fitting together "just right" to make up the total cube. If this is correct, their volume must add up to the volume of the total cube:

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.$$

Can you find the volume of each of the small pieces if the faces have the dimensions shown below?



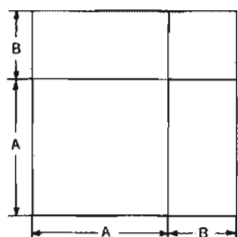
[page 112]

(23) Which identity in question 19 do these volumes suggest? (In fact, if you really "believe" this picture, it virtually gives you a **proof** of one of the identities. How come?)

(23) See the answer to question 22.

(24) If you were not sure of your answer to question 21, suppose the dimensions were written in like this:

(24) See the answer to question 21.



Compute the area of the large square (of side  $A + B$ ) by two different methods.

Now, which identity in question 19 does this suggest? (In fact, if you "believe in" this picture, you have almost "proved" the identity. How come?)

(25) Anne says that she remembers

$$(R + S)^2$$

by saying:

"The square of the first term  $R^2$

plus twice the product of the two terms  $R^2 + 2RS$

plus the square of the second term."  $R^2 + 2RS + S^2$

Can you complete this identity?

$$(A + B)^2 = A^2 + \underline{\hspace{2cm}}$$

(26) We've spent enough time looking at identities. Let's get back to work trying to solve

$$x^2 + Ax + B = W.$$

Where were we? How far had we gone?

(25)  $2AB + B^2$

(26) See the answer to question 14. At that point we had worked out a formula which allowed us to solve any quadratic equation of the form

$$(x - p)^2 = k$$

(provided we could find the square root,  $\sqrt{k}$ ).

(27) Solve

$$(x - 3)^2 = 169.$$

(28) Suppose you saw this problem:

$$x^2 - 6x + \dots = 49$$

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When you find the missing piece of paper, what number do you hope will be written on it?

Solve these equations.

(29)  $(x - 10)^2 = 9$

(30)  $(x - 1)^2 = 196$

(31)  $(x - 7)^2 = 36$

(32)  $x^2 - 16x + 64 = 81$

(27)  $(x - 3)^2 = 169$

UV:  $3 \rightarrow p$

$169 \rightarrow k$

$\sqrt{k} = \sqrt{169} = \pm 13$

$\{3 + 13, 3 - 13\}$ , or  $\{+16, -10\}$

(28) Our system depends upon the fact that, if you look at *this* number

$$x^2 - 6x + \dots = 49,$$

↑

if you take one-half of it,

$$\frac{1}{2} \times 6 = 3,$$

and if you *square* that, you should get the number that goes here

$$x^2 - 6x + \dots = 49.$$

↑

Hence, the number we should *like* to see there would be

$$3^2 = 9.$$

If you are not sure where we got this idea, try out a few examples:

$$(x + 1)^2 = x^2 + 2x + 1, \quad \frac{1}{2} \times 2 = 1, \quad 1^2 = 1$$

$$(x + 2)^2 = x^2 + 4x + 4, \quad \frac{1}{2} \times 4 = 2, \quad 2^2 = 4$$

$$(x + 3)^2 = x^2 + 6x + 9, \quad \frac{1}{2} \times 6 = 3, \quad 3^2 = 9$$

$$(x + 4)^2 = x^2 + 8x + 16, \quad \frac{1}{2} \times 8 = 4, \quad 4^2 = 16$$

$$(x + 5)^2 = x^2 + 10x + 25, \quad \frac{1}{2} \times 10 = 5, \quad 5^2 = 25$$

If you want to *prove* this in general, you can consider

$$\left(x + \frac{A}{2}\right)^2 = x^2 + Ax + \left(\frac{A}{2}\right)^2.$$

(29)  $(x - 10)^2 = 9$

UV:  $10 \rightarrow p$ ,

$9 \rightarrow k$

$\sqrt{k} = \sqrt{9} = \pm 3$

$\{10 + 3, 10 - 3\}$ , or  $\{+13, +7\}$

(30)  $(x - 1)^2 = 196$

UV:  $1 \rightarrow p$ ,  $196 \rightarrow k$

$\sqrt{k} = \sqrt{196} = \pm 14$

$\{1 + 14, 1 - 14\}$ , or  $\{+15, -13\}$

(31)  $(x - 7)^2 = 36$

UV:  $7 \rightarrow p$ ,  $36 \rightarrow k$

$\sqrt{k} = \sqrt{36} = \pm 6$

$\{7 + 6, 7 - 6\}$ , or  $\{+13, +1\}$

(32) Oh! Here's some trouble! Our paper lets us solve equations of the form

$$(x - p)^2 = k;$$

but this new equation

$$x^2 - 16x + 64 = 81$$

is not of this form!

However, following Polya's suggestions, let's see if we can turn this *new* problem into some familiar *old* problem. Can we write

$$x^2 - 16x + 64$$

in the form

$$(x - p)^2?$$

Hooray! We can! Here is the way to do it:

We take  $\frac{1}{2} \times 16$ , which is 8, then square it and that is the number we find here

$$x^2 - 16x + 64.$$

↑

Consequently,  $x^2 - 16x + 64 = (x - 8)^2$ .

We now use this identity to change the *new* problem into a familiar *old* problem:

(i)  $x^2 - 16x + 64 = 81$

(ii)  $x^2 - 16x + 64 = (x - 8)^2$  This is an identity.

(iii)  $(x - 8)^2 = 81$  PN, from line (i),  
using line (ii).

Now,

$$(x - 8)^2 = 81$$

is the type that we *do* know how to solve:

UV:  $8 \rightarrow p, 81 \rightarrow k$  (Here we are using the note  
 $\sqrt{k} = \sqrt{81} = +9$  Jerry gave to his assistant.  
 $\{8 + 9, 8 - 9\}$ , or  $\{+17, -1\}$  See question 9, page 303.)

(33)  $x^2 - 2x + 1 = 4$

(33) Similar to question 32. Use the identity  $x^2 - 2x + 1 = (x - 1)^2$  to change the equation to  $(x - 1)^2 = 4$ , which we can now solve using the formula on our paper:

UV:  $1 \rightarrow p, 4 \rightarrow k$   
 $\sqrt{k} = \sqrt{4} = +2$   
 $\{1 + 2, 1 - 2\}$ , or  $\{+3, -1\}$

(34)  $x^2 - 14x + 49 = 9$

(34)  $x^2 - 14x + 49 = 9$   
 $(x - 7)^2 = 9$   
 UV:  $7 \rightarrow p, 9 \rightarrow k$   
 $\sqrt{k} = \sqrt{9} = +3$   
 $\{7 + 3, 7 - 3\}$ , or  $\{+10, +4\}$

(35)  $x^2 - 20x + 100 = 121$

(35)  $(x - 10)^2 = 121$   
 UV:  $10 \rightarrow p, 121 \rightarrow k$   
 $\sqrt{k} = \sqrt{121} = +11$   
 $\{10 + 11, 10 - 11\}$ , or  $\{+21, -1\}$

It is important to bear in mind that these last few problems have been working smoothly because in each one, if you take the number here

$$x^2 - \underline{\quad} x + \underline{\quad} = \underline{\quad},$$

↑

divide it by 2, and square the result, then you *do* get the number here

$$x^2 - \underline{\quad} x + \underline{\quad} = \underline{\quad},$$

↑

If that were *not* so, what could we do?

Anyhow, question 36 is another easy one:

$$x^2 - 6x + 9 = 16$$

↑

$$\frac{1}{2} \times 6 = 3$$

$$3^2 = 9$$

$$x^2 - 6x + 9 = 16$$

↑

and this *is* a 9.

(36)  $x^2 - 6x + 9 = 16$

(36)  $(x - 3)^2 = 16$

UV:  $3 \rightarrow p, 16 \rightarrow k$

$$\sqrt{k} = \sqrt{16} = \pm 4$$

$$\{3 + 4, 3 - 4\}, \text{ or } \{7, -1\}$$

(37) Can you find the truth set for the open sentence

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = w?$$

(37) This, too, is of the *easy* type, since if we take the number here

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = w,$$

↑

divide it by 2, and then square this, we find that this *is* the number here

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = w.$$

↑

Hence we have

$$\left(x - \frac{a}{2}\right)^2 = w$$

and the truth set is

$$\left\{\frac{a}{2} + \sqrt{w}, \frac{a}{2} - \sqrt{w}\right\}.$$

**NOTE:**

Questions 38 through 44 present a typical Madison Project sequence. Question 38 sets the task. (If a student solves this, well and good. If not, he should go to the following problems.) Questions 39 through 44 discuss various ideas for attacking question 38. Question 44 is really a recapitulation of question 38. By *then*, students *should* be able to answer the question.

(38) Can you find the truth set for the open sentence

$$x^2 - ax + b = w?$$

(38)  $\left\{\frac{a}{2} + \sqrt{w - b + \left(\frac{a}{2}\right)^2}, \frac{a}{2} - \sqrt{w - b + \left(\frac{a}{2}\right)^2}\right\}$

If a student was able to solve question 38 on its first appearance, he presumably did something like this:

(i) Possibly

$$\left(\frac{a}{2}\right)^2 = b$$

is true, and possibly it's false. I don't know for sure.

(ii) But, in order to use the paper

$$\begin{cases} (x - p)^2 = k \\ \{p + \sqrt{k}, p - \sqrt{k}\} \end{cases}$$

I must be *sure* that, if I take the number here

$$x^2 - \underset{\uparrow}{ax} + b = w,$$

take  $\frac{1}{2}$  of it

$$\frac{a}{2}$$

and square the result

$$\left(\frac{a}{2}\right)^2,$$

this *will* be the number here

$$x^2 - ax + \underset{\uparrow}{b} = w.$$

(iii) Since I am not sure about  $b$ , I'll move the  $b$  out of the way (by subtracting  $b$  from each side of the equation):

$$x^2 - ax = w - b.$$

(iv) Since I know I *must* have

$$\left(\frac{a}{2}\right)^2$$

in the spot here

$$x^2 - ax \underset{\uparrow}{=} w - b,$$

I'll put it there (by adding it to each side of the equation):

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = w - b + \left(\frac{a}{2}\right)^2.$$

(v) Now, I can use the identity

$$x^2 - ax + \left(\frac{a}{2}\right)^2 = \left(x - \frac{a}{2}\right)^2$$

to let me rewrite the original equation as

$$\left(x - \frac{a}{2}\right)^2 = w - b + \left(\frac{a}{2}\right)^2.$$

Here we again use the note that Jerry gave to his assistant (see question 9, page 303).

(vi) But this last equation is like the one on my paper! I can solve it by merely using UV:

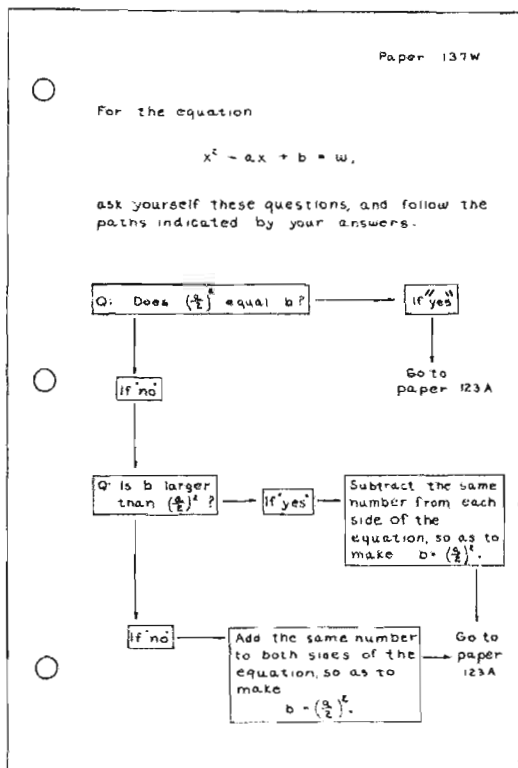
$$\begin{aligned} \text{UV: } \frac{a}{2} &\rightarrow p \\ w - b + \left(\frac{a}{2}\right)^2 &\rightarrow k \end{aligned}$$

Thus the truth set must be

$$\left\{ \frac{a}{2} + \sqrt{w - b + \left(\frac{a}{2}\right)^2}, \frac{a}{2} - \sqrt{w - b + \left(\frac{a}{2}\right)^2} \right\}.$$

This last formula represents the *general solution* for the *general quadratic equation*. (Don't worry that it looks different from the "traditional" formula. See question 1, Chapter 35.)

(39) Ellen gave her assistant a piece of paper which said:



[page 114]

What do you suppose it says on Paper 123A?

(40) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 10x + 15 = 39?$$

Trace out your path on Ellen's "map" (which mathematicians call a "flow chart").

(39)  $x^2 - ax + b = w$

Paper 123A

Use this method only if  $\left(\frac{a}{2}\right)^2 = b$ .

The original equation is equivalent to

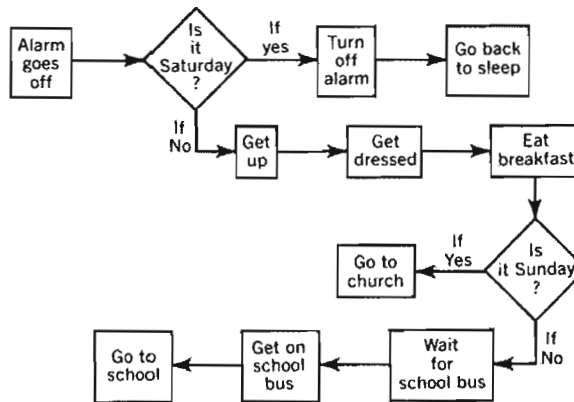
$$\left(x - \frac{a}{2}\right)^2 = w$$

and the truth set is

$$\left\{ \frac{a}{2} + \sqrt{w}, \frac{a}{2} - \sqrt{w} \right\}.$$

(40) The paper that Ellen gave to her assistant contains an example of a so-called "flow-diagram," as used in modern electronic digital computing work. Flow-diagrams of one sort or another are often found on the walls in classrooms these days; for example, here is one made up by some children:





Flow-diagrams are valuable when we want to see in very explicit form the framework of *decision making* that is involved in solving some problem. If we follow our path through Ellen's diagram, in order to solve the equation  $x^2 - 10x + 15 = 39$ , here is the result:

Q: Does  $\left(\frac{a}{2}\right)^2$  equal  $b$  ?

$x^2 - 10x + 15 = 39$   
 $\frac{1}{2} \times 10 = 5$   
 $5^2 \neq 15$

The answer is "no."

Q: Does  $\left(\frac{a}{2}\right)^2$  equal  $b$  ?

If "no"

Q: Is  $b$  larger than  $\left(\frac{a}{2}\right)^2$  ?

$a = 10$   
 $b = 15$   
 $\left(\frac{a}{2}\right)^2 = 25$   
 $15 < 25$   
 $b < \left(\frac{a}{2}\right)^2$

The answer is "no."

Q: Does  $\left(\frac{a}{2}\right)^2$  equal  $b$  ?

If "no"

Q: Is  $b$  larger than  $\left(\frac{a}{2}\right)^2$  ?

If "no"

Add the same number to both sides of the equation so as to make

$b = \left(\frac{a}{2}\right)^2$

$15 + 10 = 25$   
 $\uparrow$   
 $x^2 - 10x + 15 + 10 = 39 + 10$   
 $x^2 - 10x + 25 = 49$

Now, go to Paper 123A.

When we refer to Paper 123A, we find that it enables us to solve the equation

$$x^2 - 10x + 25 = 49$$

without difficulty:

$$\begin{aligned} (x - 5)^2 &= 49 \\ \text{UV: } 5 &\rightarrow \frac{a}{2}, \\ 49 &\rightarrow w \\ \sqrt{w} &= \sqrt{49} = 7 \\ \{5 + 7, 5 - 7\} &= \{+12, -2\} \end{aligned}$$

(41) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 12x + 45 = 18?$$

Again, trace out your path on Ellen's flow chart.

(42) Pretend that you are Ellen's assistant. How would you solve the equation

$$x^2 - 22x + 121 = 196?$$

Trace out your path on the flow chart on Paper 137W.

(43) Can Ellen's assistant solve any quadratic equation in the world?

(44) Anne and Jeanne worked out a general solution for the general quadratic equation this way:

$$\begin{aligned} x^2 - Ax + B &= W \\ x^2 - Ax &= W - B \\ x^2 - Ax + \left(\frac{A}{2}\right)^2 &= W - B + \left(\frac{A}{2}\right)^2 \\ \left(x - \frac{A}{2}\right)^2 &= W - B + \left(\frac{A}{2}\right)^2 \end{aligned}$$

What is the truth set for this open sentence? (Remember, in Descartes' words,  $x$  is the "variable" or the "unknown," whereas  $A$ ,  $B$ , and  $W$  are "constants.") Mathematicians call this method "completing the square."

(41)  $\{9, 3\}$ . Follow the same general procedure as in the answer to question 40.

(42)  $\{25, -3\}$ . Follow the same general procedure as in the answer to question 40. Notice that  $\frac{1}{2} \times 22 = 11$ ,  $11^2 = 121$ , so the answer to the first question is "yes."

(43) Yes, provided she doesn't encounter trouble in finding the square root.

(44) This is in our familiar form:

$$\begin{aligned} (x - p)^2 &= k \\ \{p + \sqrt{k}, p - \sqrt{k}\} \end{aligned}$$

Hence, we solve it by using UV:

$$\text{UV: } \frac{A}{2} \rightarrow p$$

$$W - B + \left(\frac{A}{2}\right)^2 \rightarrow k.$$

The truth set must be

$$\left\{ \frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2}, \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2} \right\}.$$

## SOME HISTORY

Schwab (1) and Eves (151), see Appendix A, provide excellent background for this chapter.



### CHAPTER 35

## Some History

[page 115]

There have been two advanced periods of civilization in European or Western history. The first was the civilization of the ancient Greeks (and their neighbors, such as Sumerians, Persians, and so forth). If we try to identify the early beginnings of this ancient civilization by using mathematics as our criterion, we might decide that it was well under way, in Babylonia, by 2000 B.C.\*

Just before 1000 B.C., the site of the "most advanced" civilization began to shift from Egypt and Babylonia to the areas inhabited by the Hebrews, Assyrians, Phoenicians, Greeks, and at least as far west as Sicily. Of this newly developing society, Howard Eves writes, "The static outlook of the ancient orient became impossible and in a developing atmosphere of rationalism men began to ask why as well as how."<sup>1</sup>

Some great mathematical thinkers of this period were Thales (around 600 B.C.), Pythagoras (born about 575 B.C.; died about 500 B.C.), Zeno (495-435 B.C.), Eudoxus (408-355 B.C.), Diophantus (born about 400 B.C.), Euclid (330-275 B.C.), Archimedes (287-212 B.C.), and Hipparchus (born about 160 B.C.). Using these men as guides, we might say that the "golden age" of Greek mathematics began roughly at 600 B.C., and began to vanish around 200 B.C. Some Greek mathematicians were still at work as late as 250 A.D. (e.g., Pappus), but by then the "golden age" was well over. Indeed, all of ancient civilization was gradually destroyed as a living society, although fragments of it remain, in various forms, even today. Human life in the Western world entered the period known as the "Dark Ages."<sup>1</sup>

(Mathematical activity was not confined solely to Europe. Indeed, very important mathematical discov-

\*Compare the interesting accounts given in: Howard Eves, *An Introduction to the History of Mathematics*, Revised Edition, pp. 30-31, and elsewhere (Holt, Rinehart, and Winston, New York, 1964).

<sup>1</sup>Ibid., p. 52.

<sup>1</sup>Compare Eves, *op. cit.*, p. 165 and Chapter 8.

## ANSWERS AND COMMENTS

eries were made by the Hindus, particularly with regard to better methods for writing mathematics. These Hindu discoveries were later to play a very important role in Western mathematical progress, but the Hindu results were largely unknown to the ancient Greeks.)

The dates for the Dark Ages may be taken as roughly 450 A.D. until 1000 A.D. Of this period, Eves writes:

The period starting with the fall of the Roman Empire in the middle of the fifth century and extending into the eleventh century is known as Europe's Dark Ages, for during this period civilization in western Europe reached a very low ebb. Schooling became almost nonexistent, Greek learning all but disappeared, and many of the arts and crafts bequeathed by the ancient world were forgotten. Only the monks of the Catholic monasteries, and a few cultured laymen, preserved a slender thread of Greek and

[page 116]

Latin learning. The period was marked by much physical violence and intense religious faith. The old social order gave way and society became feudal and ecclesiastical.

The Romans had never taken to abstract mathematics, but contented themselves with merely practical aspects of the subject associated with commerce and civil engineering. With the fall of the Roman Empire and the subsequent closing down of much of east-west trade and the abandonment of state engineering projects, even these interests waned and it is no exaggeration to say that very little in mathematics, beyond the development of the Christian calendar, was accomplished in the West during the whole of the half millennium covered by the Dark Ages.\*

Around 1000 A.D. Western society "started moving again"—started building along a "new frontier" of civilization, as we today might describe it—although for the 500 years from 1000 A.D. until 1500 A.D. the pace was fairly slow. This rebirth occurred in part as a result of Asian culture reaching Europe, particularly by way of Spain. The case of the French mathematical scholar and churchman, Gerbert, is interesting and points in the direction of what was to come. Gerbert was born in Auvergne, France, about 950 A.D. He traveled to Spain, where he studied in a Moslem school, and he may have been the person who introduced into Europe some of the Hindu methods for writing mathematics. He may also have designed clocks and musical instruments. In the year 999 he became the leader of the Roman Catholic Church, assuming the title of Pope Sylvester II.

As we have seen earlier, in the early part of the thirteenth century (just after 1200 A.D.) something new was added to European life: the great universities were started, particularly at Paris, Oxford, Cambridge, Padua, and Naples. A new civilization was beginning to appear—the one of which we, today, are the most recent part.

As history goes, our civilization is surprisingly new: even being generous, we would say that it is less than 1000 years old; looked at more narrowly, we might date it from about 1453, in which case it is more like 500 years old. Many parts of the United States are of the order of 100 years old (e.g., the state of Colorado),

\*Eves, *op. cit.*, Chapter 8.

and others (such as the city of St. Louis, Missouri) are about 200 years old. Of course, all of our "civilization" in the United States has been built upon the civilization of Europe, and in surprisingly many ways we have even built upon what we have learned from the civilizations of the ancient Greeks and their neighbors. We are nonetheless in many important respects a surprisingly new society, and we clearly have the feeling that we are headed for new frontiers—although, as always when civilization is moving forward to new and unprecedented heights, we cannot see where we are going.

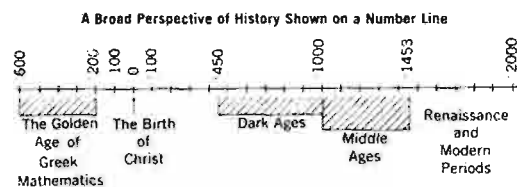
In all of the approximately 4000 years from the earliest beginnings of ancient civilization, through the Dark Ages, and up until the present time, at what point did the study of quadratic equations appear?

Clearly, were you able to show a problem in quadratic equations to a "typical" man of the Dark Ages, [page 117]

he would have been unable to solve it. Presumably he would even have been unable to understand what it was that you were trying to do. Quite likely he would not have cared, anyhow.

But the question is, was this ignorance of something that the ancients had worked on and finally came to understand, or was it ignorance of new mathematical discoveries that were not made until after the Renaissance? Had this knowledge been "lost," or had it not yet been discovered?

When did men first learn to work with quadratic equations, and to understand them?



The answer is surprising, if not nearly incredible. The study and understanding of quadratic equations is very old indeed. It dates from the early beginnings of ancient civilization. Here is what Eves writes about it:

"By 2000 B.C. Babylonian arithmetic had evolved into a well-developed rhetorical algebra. Not only were quadratic equations solved, both by the equivalent of substituting in a general formula and by completing the square, but some cubic (third degree) and biquadratic (fourth degree) equations were discussed."<sup>3</sup>

We need to clarify one point. The way that we today write quadratic equations is due to Descartes, who lived in the seventeenth century. Hence, the ancients clearly did not have our modern method of writing quadratic equations. (But, then, they also wrote num-

<sup>3</sup>Eves, *op. cit.*, p. 33

bers differently, so that does not necessarily put them out of the running for understanding quadratic equations and being able to work with them.)

It might be well for us to recapitulate the process of "completing the square." Here it is, using numbers:

$$(a) x^2 - 6x + 4 = 11$$

(b) We see that this number is not what we wish it were. Why? In fact,

$$\begin{aligned} \frac{1}{2} \times 6 &= 3 \\ 3^2 &= 9, \end{aligned}$$

and so we wish that this number were 9.

(c) Consequently, we decide to add 5 to each side of the equation, so that we have

$$x^2 - 6x + 9 = 16. \quad [\text{page 118}]$$

(d) Now, we can rewrite this as

$$(x - 3)^2 = 16.$$

(e) Evidently, the truth set for the equation in line (d) is:

$$\{7, -1\}.$$

(f) Unfortunately, we have solved the wrong problem. The set  $\{7, -1\}$  is the truth set for the equation

$$(x - 3)^2 = 16.$$

However, we were asked to find the truth set for the open sentence

$$x^2 - 6x + 4 = 11.$$

(g) Fortunately, the two changes in the equation which we have made were both "transform operations." Hence, the equation

$$x^2 - 6x + 4 = 11$$

has exactly the same truth set as the equation

$$(x - 3)^2 = 16.$$

(h) Hence, the truth set for the equation

$$x^2 - 6x + 4 = 11$$

is

$$\{7, 1\}.$$

If we use Descartes notion of "constants" and "variables," we can use this same method of "completing the square" to derive the general solution of the general quadratic equation.

(a) The general quadratic equation can be written in the form

$$x^2 - Ax + B = W.$$

(b) Now, we want to make sure that

$$\left(\frac{1}{2} \times A\right)^2 = B.$$

Since we do now know, in general, whether or not this is true, we can avoid the matter entirely by subtracting  $B$  from each side of the equation:

$$x^2 - Ax = W - B.$$

(c) Now, since we want to see

$$\left(\frac{A}{2}\right)^2$$

inserted on the left-hand side, our simplest procedure will be to put it there. How can we do this in a legal

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fashion (that is, by using a "transform operation")? The answer is simple: we shall add

$$\left(\frac{A}{2}\right)^2$$

to each side of the equation, which gives us:

$$x^2 - Ax + \left(\frac{A}{2}\right)^2 = W - B + \left(\frac{A}{2}\right)^2.$$

(d) Since we have now made the left-hand side into a "perfect square," we can write:

$$\left(x - \frac{A}{2}\right)^2 = W - B + \left(\frac{A}{2}\right)^2.$$

(e) For this equation, the truth set is

$$\left\{ \frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2}, \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2} \right\}.$$

We can now hand our assistant a piece of paper that says:

For the open sentence

$$x^2 - Ax + B = W,$$

the truth set is

$$\left\{ \frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2}, \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2} \right\}.$$

This is a secret from antiquity. Use it in good health!

(1) Many books in the first half of the twentieth century wrote the general quadratic equation in the form

$$ax^2 + bx + c = 0.$$

Use the formula which we just obtained to find the truth set for this equation.

Although the ancients solved the general quadratic equation, and also some cubic and quartic equations,

- (1) Starting with the form  $ax^2 + bx + c = 0$ , we know that  $a \neq 0$  (for otherwise the equation would be *linear* and *not quadratic*). Consequently, we may divide both sides of the equation by  $a$ , getting

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \quad (1)$$

This is *almost* our form

$$x^2 - Ax + B = W. \quad (2)$$

they were never able to solve the general cubic equation nor the general quartic equation. In large part they must have been handicapped by their lack of our modern methods for writing mathematics. (Imagine doing mathematics, for example, with your eyes closed!)

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When did man first come to understand the general cubic and quartic equations? Here is what Eves writes, describing the event:

Probably the most spectacular mathematical achievement of the sixteenth century was the discovery, by Italian mathematicians, of the algebraic solution of cubic and quartic equations. The story of this discovery, when told in its most colorful version, rivals any page written by Benvenuto Cellini. Briefly told the facts seem to be these. About 1515, Scipione del Ferro (1465-1526), a professor of mathematics at the University of Bologna, solved algebraically the cubic equation  $x^3 + mx = n$ , probably basing his work on earlier Arabic sources. He did not publish his result but revealed the secret to his pupil Antonio Fior. Now about 1535, Nicolo of Brescia, commonly referred to as Tartaglia (the stammerer) because of a childhood injury which affected his speech, claimed to have discovered an algebraic solution of the cubic equation  $x^3 + px^2 = n$ . Believing this claim was a bluff, Fior challenged Tartaglia to a public contest of solving cubic equations, whereupon the latter exerted himself and only a few days before the contest found an algebraic solution for cubics lacking a quadratic term. Entering the contest equipped to solve two types of cubic equations, whereas Fior could solve but one type, Tartaglia triumphed completely. Later Girolamo Cardano, an unprincipled genius who taught mathematics and practiced medicine in Milan, upon giving a solemn pledge of secrecy wheedled the key to the cubic from Tartaglia. In 1545, Cardano published his *Ars magna*, a great Latin treatise on algebra, at Nuremberg, Germany, and in it appeared Tartaglia's solution of the cubic. Tartaglia's vehement protests were met by Lodovico Ferrari, Cardano's most capable pupil, who argued that Cardano had received his information from del Ferro through a third party and accused Tartaglia of plagiarism from the same source. There ensued an acrimonious dispute from which Tartaglia was perhaps lucky to escape alive.

Since the actors in the above drama seem not always to have had the highest regard for truth, one finds a number of variations in the details of the plot.

The solution of the cubic equation  $x^3 + mx = n$  given the Cardano in his *Ars magna* is essentially the following. Consider the identity

$$(a - b)^3 + 3ab(a - b) = a^3 - b^3.$$

If we choose  $a$  and  $b$  so that

$$3ab = m, \quad a^3 - b^3 = n,$$

then  $x$  is given by  $a - b$ . Solving the last two equations simultaneously for  $a$  and  $b$  we find

$$a = \sqrt[3]{\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 - \left(\frac{m}{3}\right)^3}}$$

$$b = \sqrt[3]{-\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 - \left(\frac{m}{3}\right)^3}}$$

and  $x$  is thus determined.

It was not long after the cubic had been solved that an algebraic solution was discovered for the general quartic (or biquadratic) equation. In 1540, the Italian mathematician Zuanne de Tonini da Coi proposed a problem to Cardano which led to a quartic equation. Although Cardano was unable to solve the equation, his pupil Ferrari succeeded, and Cardano had the pleasure of publishing this solution also in his *Ars magna*.\*

(2) Explain Cardano's solution of the equation

$$x^3 + mx = n.$$

If we rewrite equation (1) as

$$x^3 - \frac{b}{a}x + \frac{c}{a} = 0,$$

then we can solve it by using equation (2) and UV.

$$UV: \frac{b}{a} \rightarrow A, \frac{c}{a} \rightarrow B, 0 \rightarrow W.$$

Then, the truth set

$$\left\{ \frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2}, \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2} \right\}$$

becomes

$$\left\{ \frac{b}{2a} + \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}}, \frac{b}{2a} - \sqrt{-\frac{c}{a} + \frac{b^2}{4a^2}} \right\}.$$

We can put these over a common denominator of  $2a$ , getting

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\},$$

and, if we write this in traditional notation, we get the traditional formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note: There are good reasons for avoiding the "traditional" notation  $\pm$ . This notation has been used ambiguously in the past. Sometimes it has meant that either sign might be chosen and would necessarily be correct. At other times it has been used to mean that one sign or the other was correct, but not necessarily both.

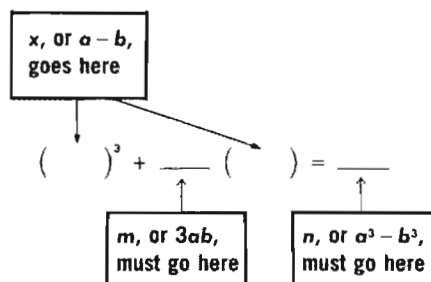
(2) Whether by good luck, or otherwise, Cardano had an opportunity to think about the identity

$$(a - b)^3 + 3ab(a - b) = a^3 - b^3 \quad (1)$$

\*Eves, *op. cit.*, pp. 220-221.



The similarity of the two forms which Tartaglia perhaps observed is illustrated in the diagram below:



and to observe its similarity to the equation he wished to solve,

$$x^3 + mx = n. \quad (2)$$

There is a striking similarity of form between the two; each involves some number cubed,

$$( \quad )^3,$$

plus this same number to the first power, with some coefficient,

$$( \quad )^3 + \dots ( \quad ),$$

set equal to something else,

$$( \quad )^3 + \dots ( \quad ) = \dots$$

If we want to solve equation (2), can we first turn it into equation (1)? This would require

$$\begin{aligned} x &= a - b \\ m &= 3ab \\ n &= a^3 - b^3 \end{aligned}$$

But this works out very nicely! If we are given the equation

$$x^3 + mx = n,$$

that means we are given  $m$  and  $n$ , and asked to find  $x$ . But if we are given  $m$  and  $n$ , then the equations

$$\begin{aligned} m &= 3ab \\ n &= a^3 - b^3 \end{aligned}$$

can be used to determine  $a$  and  $b$  (we shall work this out in just a moment).

Now, since we know  $a$  and  $b$ , we can determine  $x$ ,

$$x = a - b,$$

which completes the job. (Well, there might be some *other* possible values of  $x$  that we have overlooked, but at least we have made a beginning.)

Perhaps we should go backwards a moment, and see how we got from  $m$  and  $n$  to  $a$  and  $b$ . We had

$$m = 3ab \quad (3)$$

$$n = a^3 - b^3 \quad (4)$$

If  $m = 0$ , we know  $x = \sqrt[3]{n}$ . Hence, assume  $m \neq 0$ . Then  $a \neq 0$  and  $b \neq 0$ , and we can divide both sides of equation (3) by  $3b$ , getting

$$a = \frac{m}{3b},$$

which can then be used to eliminate  $a$  from equation (4).

The rest of the work is straightforward, provided you are careful with radicals and exponents.

(3) Cardano, in effect, had written this note:

$$\begin{aligned} &\text{For the equation} \\ &x^3 + mx = n, \\ &x = a - b, \text{ where} \\ &a = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} \\ &b = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} \end{aligned}$$

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(Today we know that there may be other solutions for this equation, but Cardano's work is correct as far as it goes.)

Suppose Cardano met the equation

$$x^3 + ax^2 + bx + c = 0.$$

What would he do?

[Hint: He would use Polya's idea of "reducing it to some other problem that he already knew how to solve." One way to do this is to write

$$x = t + \alpha.]$$

Can you finish this problem?

(3) If we let

$$x = t + \alpha,$$

and substitute into the equation

$$x^3 + ax^2 + bx + c = 0,$$

we get

$$(t + \alpha)^3 + a(t + \alpha)^2 + b(t + \alpha) + c = 0$$

which is

$$t^3 + 3t^2\alpha + 3\alpha^2t + \alpha^3 + at^2 + 2at\alpha + a\alpha^2 + bt + b\alpha + c = 0$$

$$t^3 + (3\alpha + a)t^2 + (3\alpha^2 + 2a\alpha + b)t + (\alpha^3 + a\alpha^2 + b\alpha + c) = 0$$

We now (having looked carefully at this last equation above) select  $\alpha$  such that

$$3\alpha + a = 0;$$

that is, so that  $\alpha = -\frac{a}{3}$ , thereby getting a cubic equation for  $t$  in which there is no  $t^2$  term. We can then use Cardano's procedure.



## Part Five ■ Matrices

chapter 36 / Pages 122-130 of Student Discussion Guide

### THE IDEA OF “MAPPINGS” OR “CORRESPONDENCES”

In a formal sense, *mapping*, *correspondence*, and *function* all refer to the same thing. Psychologically, however, they do not, and the way we *think* about mathematics is every bit as important as the way we write it or explain it.



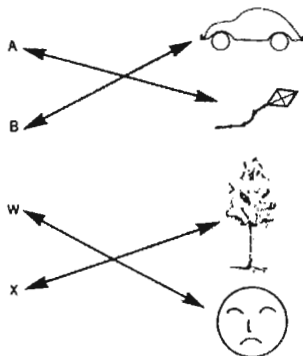
#### CHAPTER 36

### The Idea of “Mappings” or “Correspondences”

[page 122]

Mathematicians nowadays think that the idea of correspondences is very important. Let's see if we can figure out what they mean when they use the word correspondence.

Jerry made this correspondence:



(1) In Jerry's scheme, what corresponds to *A*?

*A* ↔ \_\_\_\_\_

(2) In Jerry's scheme, what corresponds to *W*?

*W* ↔ \_\_\_\_\_

(3) In Jerry's scheme, what corresponds to



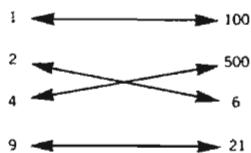
### ANSWERS AND COMMENTS

(1) ***A* corresponds to the kite.**

(2) ***W* corresponds to the face.**

(3) **The car corresponds to *B*.**

(4) Andy made this correspondence:



In Andy's scheme, what corresponds to 1?

(5) In Andy's scheme, what corresponds to 2?

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(6) Suppose that  $A$  is this set:



And suppose that  $B$  is this set:



Can you make a correspondence between the elements of  $A$  and the elements of  $B$ ?

In your scheme, which element of  $B$  corresponds to



Will others in your class have a correspondence different from yours?

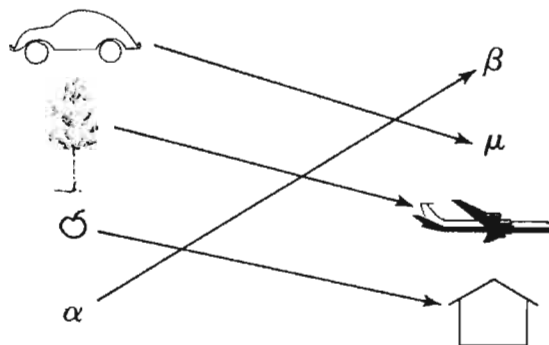
Which correspondence in your class is correct?

(4) 100

(5) 6

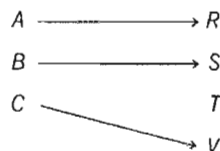
(6) This will depend upon your class, and each student's correspondence may, indeed, be different and correct.

The correspondence need not be one-to-one. For example, here is a one-to-one correspondence:



Every element of set  $A$  corresponds to exactly one element of set  $B$ , and every element of set  $B$  corresponds to exactly one element of set  $A$ . (This is often called "one-to-one onto.")

Let's look at what mathematicians mean when they say a mapping is "one-to-one into." Let set  $W$  be  $\{A, B, C\}$  and let set  $U$  be  $\{R, S, T, V\}$ . Then the following mapping is "one-to-one into":



To say it more fully, we have "mapped set  $W$  one-to-one into set  $U$ ." Since set  $W$  contains three elements, and set  $U$  contains four elements, we cannot possibly map set  $W$  one-to-one onto set  $U$ .

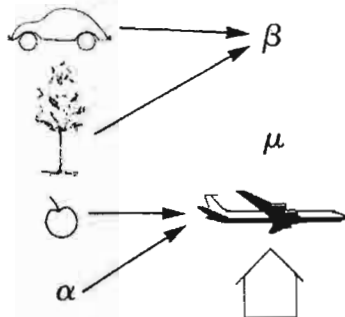
We could describe the idea of a "one-to-one into" mapping by saying:

Set  $X$  is mapped one-to-one into set  $Y$  if every element of set  $X$  corresponds to exactly one element of set  $Y$ , while every element of set  $Y$  either corresponds to exactly one element of set  $X$ , or else to no element of set  $X$ .

The mapping

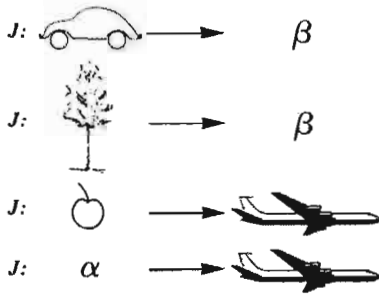


(7) Joe says this is a mapping of the set  $A$  into the set  $B$ :



What do you think?

(8) Tom wrote Joe's mapping like this:



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Can you make up another mapping of the set  $A$  into the set  $B$ ? Can you write it the way Joe wrote his? Can you write it using Tom's method?

(9) How many different mappings of set  $A$  into set  $B$  can you find? How many different mappings of  $A$  into  $B$  do you suppose there are altogether?

is *not* one-to-one because the element  $R$  corresponds both to  $A$  and to  $B$ .

You need not worry about these ideas at all at this point, unless you wish to.




(7) **Joe is right.** We say we have mapped set  $A$  into set  $B$  if every element of  $A$  corresponds to exactly one element of  $B$ , but not necessarily vice versa. This, then, is a mapping of  $A$  into  $B$ . We use the more restrictive phrase "mapping  $A$  onto  $B$ " if every element of  $B$  is the image of at least one element of  $A$ —that is to say, "all of  $B$  is covered."

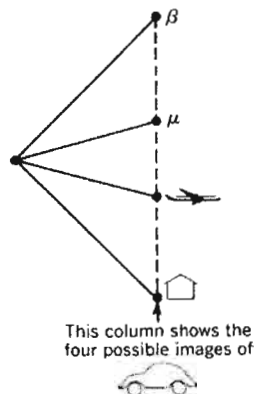
In the present example, the mapping is *not* "onto," since  $\mu$  and  $\square$  are omitted—they are not the images of any elements of  $A$ .

(8) **Tom's notation is the most important idea in this chapter.** I hope it is clear how this notation works. (If not, wait until you have seen a few more examples.)

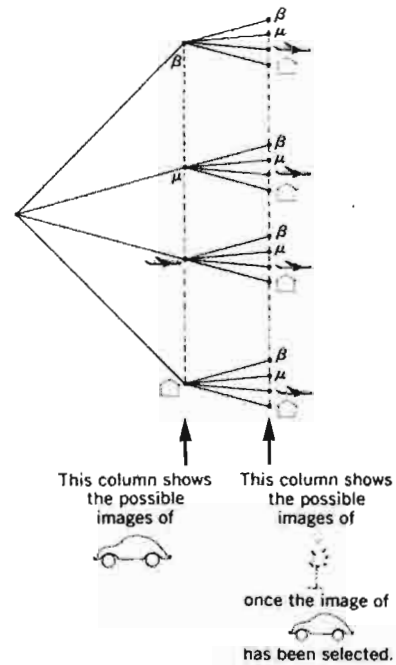
(9) **There are 256 different mappings.**

Let's try to count them:

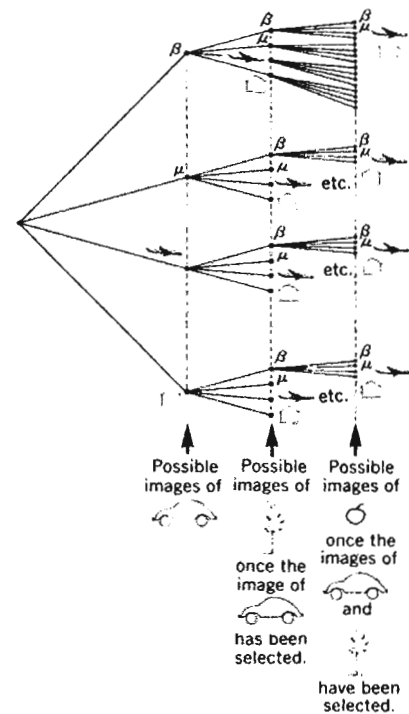
(i) We can map  in four different ways (into  $\beta$ ,  $\mu$ , , or ). Perhaps, to keep track of all the possibilities, we should make a *tree diagram*, as we did in Chapter 14.



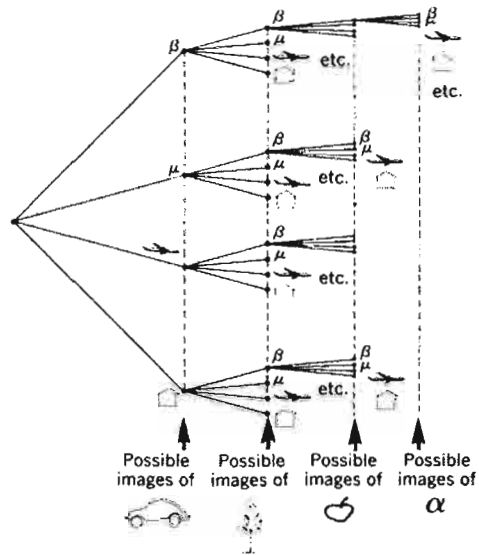
(ii) After choosing one of these four possibilities, we face a new decision point (as the tree diagram will show). We can map into any of four different images (remember, we are *not* requiring the mapping to be one-to-one):



(iii) After making our second choice (namely, how to map  $\circ$ ), we again come to a new decision point, and again have four choices for mapping  $\circ$ :



(iv) Finally, for each choice of the image of  $\mathcal{O}$ , there are four possible ways to map  $\alpha$  :



Now, we claim that every possible path through this “maze” (or tree diagram), remembering that each arm is a one-way street, corresponds to exactly one mapping of the set  $A$  into the set  $B$ ; also, conversely, every mapping of  $A$  into  $B$  corresponds to a trip through this “maze.” There are evidently

$$4^4 = 256$$

“exits” from the maze, which (remembering that each arm is a one-way street) means there are 256 different paths through the maze, which means there are 256 different ways to map  $A$  into  $B$ .

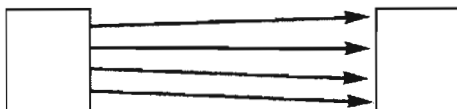
(10) Al says one way to show a mapping of set  $A$  into set  $B$  is to list the elements of  $A$  in a column here

Elements of $A$
-----------------------

and to list the elements of set  $B$  in a column here

Elements of $B$
-----------------------

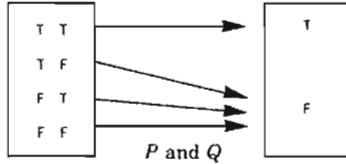
and then to draw arrows from each element of  $A$  to some of the elements of  $B$ , sort of like this:



Can you make up a mapping and write it this way?

(10) This will depend upon your class.

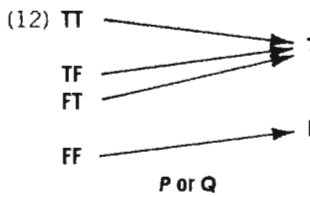
(11) Sam used Al's method to write " $P$  and  $Q$ " as a mapping of  $\{TT, TF, FT, FF\}$  into  $\{T, F\}$  like this:



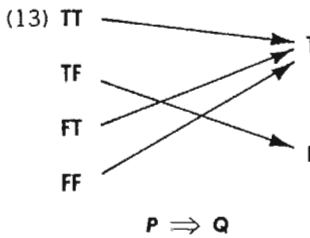
What do you think?

(12) Can you use Al's method to write " $P$  or  $Q$ " as a mapping of  $\{TT, TF, FT, FF\}$  into  $\{T, F\}$ ?

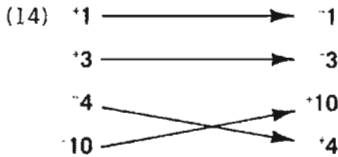
(11) **Sam is correct.**



(13) Can you use Al's method to write  $P \Rightarrow Q$  as a mapping of  $\{TT, TF, FT, FF\}$  into  $\{T, F\}$ ?

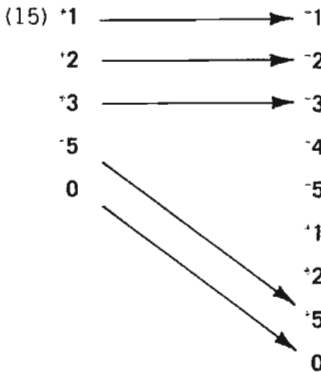


(14) Can you use the operation  $U : \square \rightarrow \circ \square$  to map the set  $\{^1, ^3, ^4, ^{10}\}$  into the set  $\{^1, ^3, ^{10}, ^4\}$ ? Can you write the mapping by Al's method?



(15) Can you use the operation  $U : \square \rightarrow \circ \square$  to map the set  $\{^1, ^2, ^3, ^5, 0\}$  into the set  $\{^1, ^2, ^3, ^4, ^5, ^1, ^2, ^5, 0\}$ ? Can you use Al's method to write this mapping?

[page 125]

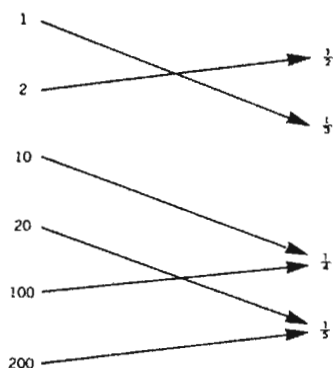


(16) Can you use the operation  $U : \square \rightarrow \circ \square$  to map the set  $\{^1, ^2, ^3, ^5\}$  into the set  $\{^1, 0, ^2, ^4\}$ ? Can you use Al's method to write this mapping?

(16) **No.** The "image" set  $\{^1, 0, ^2, ^4\}$  does not provide for an image of  $^3$  and an image of  $^5$  under the mapping  
 $U : ^3 \rightarrow ^3$   
 $U : ^5 \rightarrow ^5$



(17) Can you take the mapping



and write it, using Tom's method?

**Mappings** appear in many different disguises. Mathematicians try to see through these disguises and recognize the mapping, whenever they can.

(18) Tony says that the "Guessing Functions" game is really a mapping in disguise. What do you think? Can you make up a rule and show how it can be written using Al's method?

(19) Elizabeth says that hanging weights on a spring gives you a mapping. What do you think?

(20) Toby says that using a magnifying glass gives you a mapping of "pictures" into "large pictures." Suppose you used a magnifying glass on these pictures:



What would you get? Can you write this, using Al's method?

(21) Ellen says that you can use equations like

$$(\square \times \square) - (\square \times \square) + \square = 0$$

to map the ordered pair (5, 6) into the nonordered pair (2, 3). Can you figure out how she does it?

(17) We need to choose some letter to stand for the "rule" or "mapping" itself. Let's use  $R$ . Then we can write:

$$R: 1 \rightarrow \frac{1}{3}$$

$$R: 2 \rightarrow \frac{1}{2}$$

$$R: 10 \rightarrow \frac{1}{4}$$

$$R: 20 \rightarrow \frac{1}{5}$$

$$R: 100 \rightarrow \frac{1}{4}$$

$$R: 200 \rightarrow \frac{1}{5}$$

Tom's method was presented in problem 8, page 327.

Incidentally, we could also write this mapping using the function notation from Chapter 28. Again, we must choose a letter to stand for the rule itself. Suppose we choose  $r$ . Then we would write:

$$r(1) = \frac{1}{3} \quad (\text{Read: } r \text{ of } 1 \text{ equals one-third.})$$

$$r(2) = \frac{1}{2} \quad (\text{Read: } r \text{ of } 2 \text{ equals one-half.})$$

$$r(10) = \frac{1}{4} \quad (\text{Read: } r \text{ of } 10 \text{ equals one-fourth.})$$

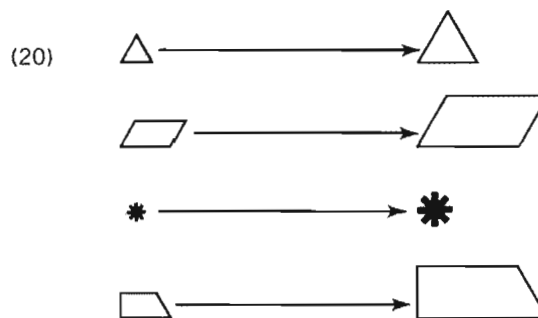
$$r(20) = \frac{1}{5} \quad (\text{Read: } r \text{ of } 20 \text{ equals one-fifth.})$$

$$r(100) = \frac{1}{4} \quad (\text{Read: } r \text{ of } 100 \text{ equals one-fourth.})$$

$$r(200) = \frac{1}{5} \quad (\text{Read: } r \text{ of } 200 \text{ equals one-fifth.})$$

(18) This will depend upon your class.

(19) Yes, it does.



(21) Ellen is correct. She puts the *ordered pair* (5, 6) into the equations as coefficients (Descartes' "constants"),

$$(\square \times \square) - (5 \times \square) + 6 = 0,$$

and then finds the truth set

$$\{2, 3\}.$$

Why does *order* make a difference for the pair (5, 6), when we use the numbers this way? Why does *order* not make a difference for (2, 3)?

[page 126]

(22) Using Ellen's method, the ordered pair (8, 15) would map into \_\_\_\_\_.

Since Ellen maps (5, 6) into (2, 3),

$$(5, 6) \longrightarrow (2, 3),$$

mathematicians say that

"(2, 3) is the image of (5, 6)."

(23) When you use Ellen's mapping, what is the image of (8, 15)?

(24) When you use Ellen's mapping, what is the image of (9, 14)?

(25) When you use Ellen's mapping, what is the image of (13, 22)?

(26) Can you write Ellen's mapping, using Al's method?

(27) The world globe is (approximately) a sphere. How could you make a flat map of the world?

The notation we are using here works as follows: where order is important, we use parentheses; wherever order is *not* important, we use braces.

$$(22) (\square \times \square) - (8 \times \square) + 15 = 0 \\ \{5, 3\}$$

So Ellen's mapping, thus far, might be written

$$E: (5, 6) \longrightarrow \{2, 3\}$$

$$E: (8, 15) \longrightarrow \{5, 3\}$$

What would *this* be:  $E: (6, 5) \longrightarrow ?$

Could we say  $E: (8, 15) \longrightarrow \{3, 5\}$ ? Yes, because it does not matter which root of an equation we say first. Hence, in this case, order is not important.

(23)  $\{5, 3\}$  is the image of (8, 15).

Or, we could also say  $\{3, 5\}$  is the image of (8, 15). [But you must not change (8, 15) to (15, 8)!]

$$(24) (\square \times \square) - (9 \times \square) + 14 = 0 \\ \{2, 7\}$$

$\{2, 7\}$  is the image of (9, 14).

$$(25) (\square \times \square) - (13 \times \square) + 22 = 0 \\ \{11, 2\}$$

$\{11, 2\}$  is the image of (13, 22).

Could we also say  $\{2, 11\}$  is the image of (13, 22)?

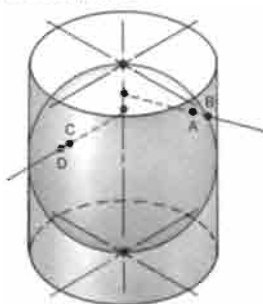
$$(26) \begin{array}{l} (5, 6) \longrightarrow \{2, 3\} \\ (8, 15) \longrightarrow \{5, 3\} \\ (6, 5) \longrightarrow \{5, 1\} \\ (9, 14) \longrightarrow \{7, 2\} \\ (13, 22) \longrightarrow \{11, 2\} \end{array}$$

We could also write Ellen's mapping, using functional notation:

$$\begin{array}{l} E(5, 6) = \{2, 3\} \\ E(8, 15) = \{5, 3\} \\ E(6, 5) = \{5, 1\} \\ E(9, 14) = \{7, 2\} \\ E(13, 22) = \{11, 2\} \end{array}$$

(27) There are many possible methods.

(28) Lex used this method:



He mapped point  $A$  (on the sphere) onto point  $B$  (on the cylinder). He mapped point  $C$  (on the sphere) onto point  $D$  (on the cylinder).

What will Lex's flat map look like? Will a country be the same size on the flat map as it is on the sphere? If not, will it be larger or smaller on the cylinder than on the sphere?

(29) John says that when we use exponents we are using a mapping in disguise.

John wrote:

$$E : \square \rightarrow 10^{\square}$$

Can you find the image of 2, using John's mapping?

(28) The dimension of a country from north to south will be greater on the sphere than it is on the cylinder; by contrast, the dimension of a country from east to west will be greater on the cylinder than it is on the sphere. Countries near the north or south pole are affected more in both of these ways than countries near the equator. (Indeed, right on the equator the east-west dimension is not changed at all.)

If we start with a country on the sphere and consider its image on the cylinder, we see that the image has been enlarged in the east-west direction, and diminished in the north-south direction. What, then, has happened to its area? Depending on the precise amount of these two changes, the area might have become larger, or smaller, or stayed the same. A careful use of similar triangles and trigonometry—an argument that we would not use below the high school level—enables you to show that, to a first approximation, the area of a country is not changed.

(29)  $E: 2 \rightarrow 100$  (Read:  $E$  maps 2 into 100.)

$$E: 3 \rightarrow 1000$$

$$E: 1 \rightarrow 10$$

$$E: 0 \rightarrow 1$$

$$E: -1 \rightarrow \frac{1}{10}$$

$$E: -2 \rightarrow \frac{1}{100}$$

Or, using functional notation:

$$E(2) = 100 \quad (\text{Read: } E \text{ of } 2 \text{ equals } 100.)$$

$$E(3) = 1000 \quad (\text{Read: } E \text{ of } 3 \text{ equals } 1000.)$$

⋮

The image of 2 is 100.

[page 127]  
(30) Use John's mapping on the set  $\{1, 2, 3, 4\}$ . What is the image set?

$$(30) E(1) = 10$$

$$E(2) = 100$$

$$E(3) = 1000$$

$$E(4) = 10,000$$

Hence, the image set is  $\{10, 100, 1000, 10,000\}$ .

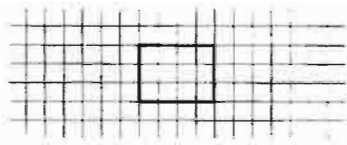
(31) Can you use Al's method to write the mapping in question 30?

- (31)  $E: 1 \rightarrow 10$  (Read:  $E$  maps 1 into 10.)  
 $E: 2 \rightarrow 100$  (Read:  $E$  maps 2 into 100.)  
 $E: 3 \rightarrow 1000$  (Read:  $E$  maps 3 into 1000.)  
 $E: 4 \rightarrow 10,000$  (Read:  $E$  maps 4 into 10,000)

(32) Using John's mapping, what is the image of 5?

(32)  $E: 5 \rightarrow 100,000$ ; so, the image of 5 is 100,000.

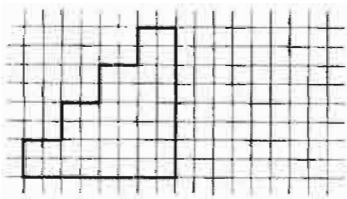
(33) Bill mapped the figure



(33) 40

into the number 12.

Using Bill's mapping, can you find the image of the following figure?



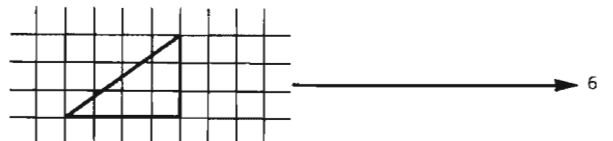
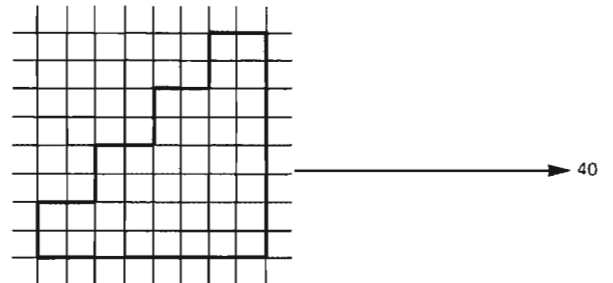
(34) Using Bill's mapping, can you find the image of this figure?

(34) 6



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(35) Can you write Bill's mapping, using Al's method?



(36) Debbie says that Bill mapped plane figures into numbers, by using the idea of area. She says she will map plane figures into segments, by drawing the figure on a grid, pretending the sun is directly overhead, and mapping the figure into its shadow on the  $x$ -axis. Debbie is pretending that the  $x$ -axis is the ground.

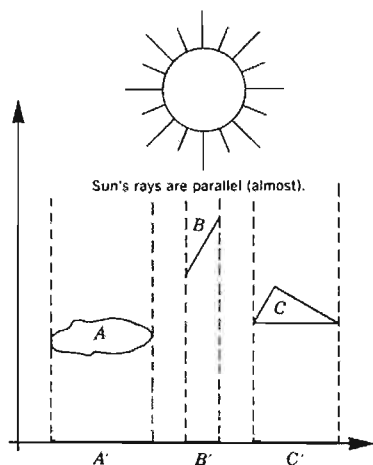
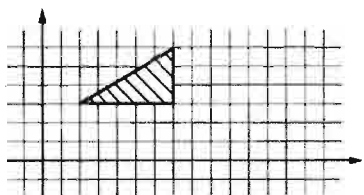


Figure A is mapped into its shadow  $A'$ .  
 Figure B is mapped into its shadow  $B'$ .  
 Figure C is mapped into its shadow  $C'$ .

Using Debbie's mapping, can you find the image of this triangle?



(37) Using Debbie's mapping, choose some figures of your own, and see if you can find the "shadows."

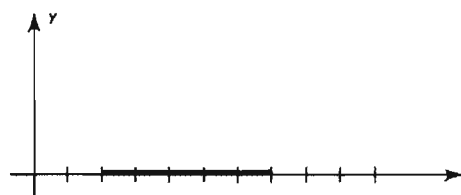
(38) Using Debbie's mapping, what is the image of the point (3, 4)?

(39) Suppose  $0 < b$ . Using Debbie's mapping, what is the image of the point  $(a, b)$ ?

(40) Suppose  $b < 0$ . Using Debbie's mapping, what is the image of the point  $(a, b)$ ?

(41) Can you use a light bulb or a flashlight to map some physical objects into their shadows?

(36)



The image is the segment

$$\{x \mid 2 \leq x \leq 7\}.$$

(Read: The set of all points with the name  $x$ , such that  $2 \leq x \leq 7$ ; or more briefly, the set of all  $x$  such that  $2 \leq x \leq 7$ .)

(37) This will depend upon your class.

(38) The image of (3, 4) is (3, 0).

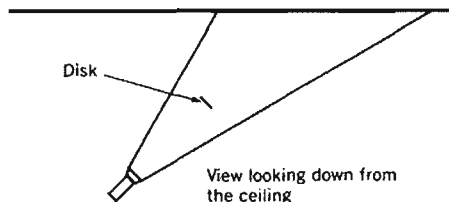
(39) If  $0 < b$ , the image of  $(a, b)$ , under Debbie's mapping, is  $(a, 0)$ . Since Debbie's mapping is called a "projection," we might represent it by the letter  $P$ . We could then write

$$P: (a, b) \rightarrow (a, 0), \text{ if } 0 < b.$$

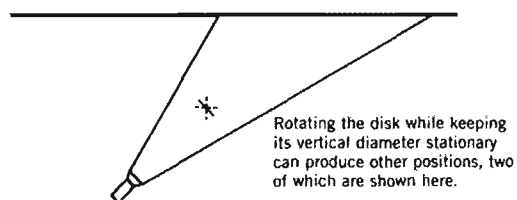
(40) Debbie's rule does not work clearly for the point  $(a, b)$ , if  $b < 0$ . Can you extend it so that it will? (Of course, there are many possible ways to extend it.)

(41) This can be fun. In fact, there is a great deal of mathematics that can be studied through shadow pictures, as shown in the following example.

Aim a flashlight obliquely at the wall. Hold a circular disk in the light, so that it is parallel to the flashlight lens (i.e., perpendicular to the center line for the cone of light).



What shape will the shadow on the wall be?  
Turn the circular disk.



Prove, mathematically, that there must be two different positions of the disk which will make the shadow on the wall circular. (One argument you can use to prove this depends upon continuity, and is essentially topological!)

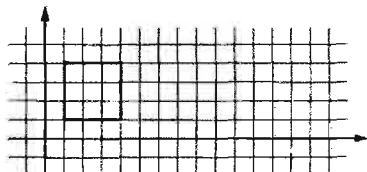
[page 129]

(42) Earle says you can map the plane (which mathematicians write " $E_2$ ") into itself.

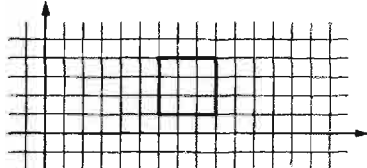
Earle made up this mapping:

$$\begin{aligned} X_{\text{new}} &= X_{\text{old}} + 5 \\ Y_{\text{new}} &= Y_{\text{old}} \end{aligned}$$

He used his mapping to transform the figure

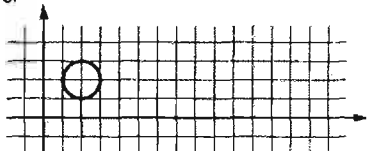


into this figure:



What do you think?

(43) Dexter used Earle's mapping to find the image of this figure.

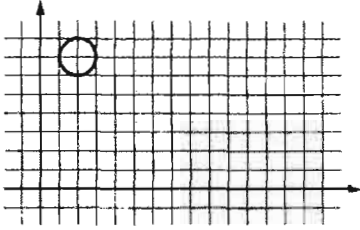


(42) Earle's mapping moves any figure 5 units to the right, but leaves it otherwise unchanged. You can easily verify this for yourself by using the numbers that are the coordinates of key points, such as (perhaps) the vertices.

(43) Dexter is wrong. He moved the figure five units up instead of five units to the right. (See answer to question 42.) To understand this you may wish to notice that the original center of the circle is located at (2, 2). Under Earle's mapping, this becomes (2 + 5, 2), which is (7, 2).

He got:

[page 130]



Do you agree? Can you draw a figure in  $E_2$  and then find its image, using Earle's mapping?

(44) Bernie mapped  $E_2$  into  $E_2$ , using this mapping:

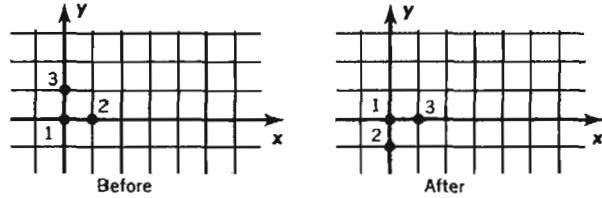
$$\begin{aligned} X_{\text{new}} &= Y_{\text{old}} \\ Y_{\text{new}} &= X_{\text{old}} \end{aligned}$$

Using Bernie's mapping, what is the image of  $(1, 0)$ ? What is the image of  $(0, 0)$ ? What is the image of  $(0, 1)$ ? Can you show this by Al's method (as Al did in question 10, earlier in this chapter)? If you start with the set  $\{(1, 0), (0, 0), (0, 1)\}$ , what is the image set?

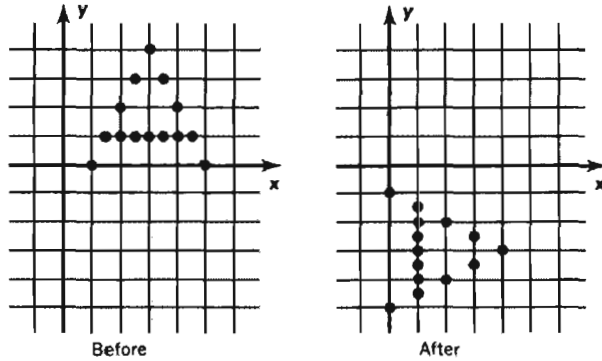
(44) Again, let's try a few points, and see if we can discover how this mapping works. Let's call the mapping  $B$ .

$$\begin{aligned} B: (0, 0) &\longrightarrow (0, 0) \\ B: (1, 0) &\longrightarrow (0, -1) \\ B: (0, 1) &\longrightarrow (1, 0) \end{aligned}$$

The image set of  $\{(1, 0), (0, 0), (0, 1)\}$  is  $\{(0, -1), (0, 0), (1, 0)\}$ .



... and the mapping  $B$  seems to rotate everything  $90^\circ$  in a clockwise direction.



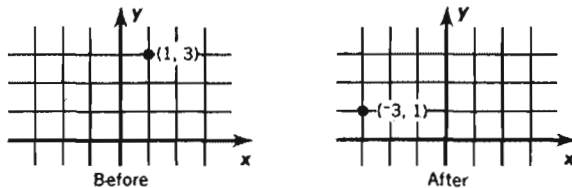
Here, again, you can verify this by performing the arithmetic on the coordinates of a few selected points.

(45) Nancy mapped  $E_2$  into  $E_2$  by this mapping:

$$\begin{aligned} X_{\text{new}} &= Y_{\text{old}} \\ Y_{\text{new}} &= X_{\text{old}} \end{aligned}$$

Can you draw a figure in  $E_2$  and then find its image, using Nancy's mapping?

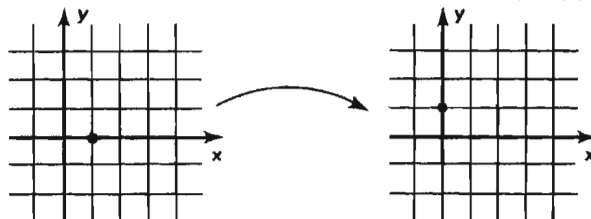
(45) Let's try a few points, first.



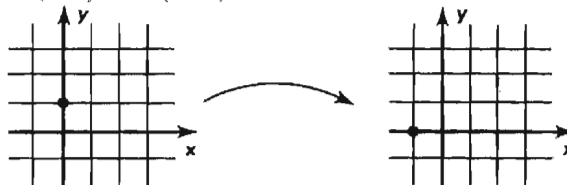
Here are some more:

$N: (0, 0) \rightarrow (0, 0)$  (Read:  $N$  maps  $(0, 0)$  into  $(0, 0)$ .)

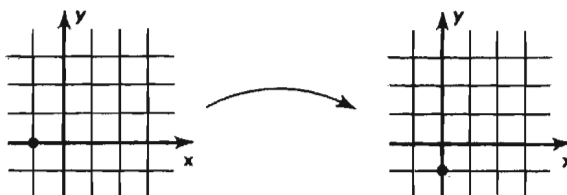
$N: (1, 0) \rightarrow (0, 1)$  (Read:  $N$  maps  $(1, 0)$  into  $(0, 1)$ .)



$N: (0, 1) \rightarrow (-1, 0)$

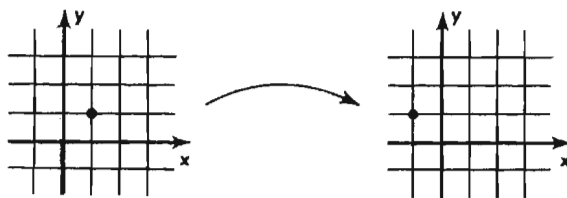


$N: (-1, 0) \rightarrow (0, -1)$

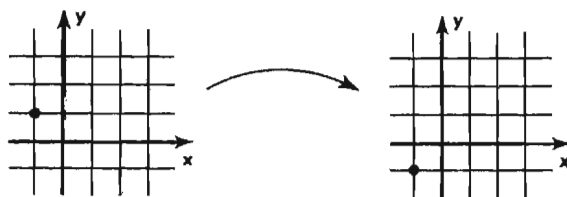


Can you give a geometrical description of what seems to be happening?

$N: (1, 1) \rightarrow (-1, 1)$

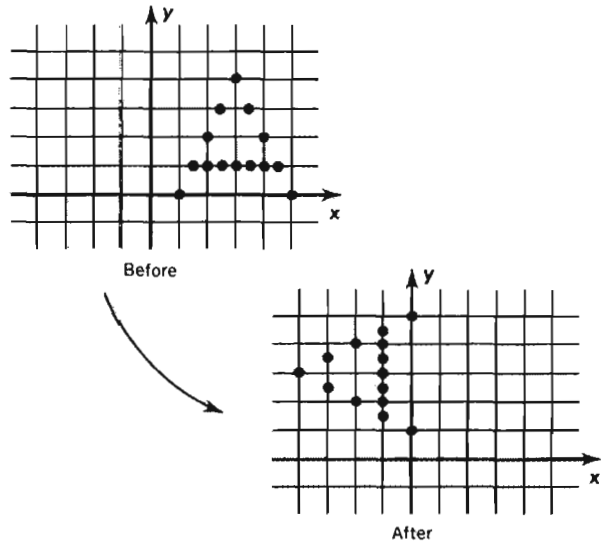


$N: (-1, 1) \rightarrow (-1, -1)$





The mapping  $N$  rotates any figure  $90^\circ$  counterclockwise. The point  $(0, 0)$  remains fixed. For example:



(46) Draw some figure in  $E_2$ . Can you find its image using Bernie's mapping?

(46) See answer to question 44.

(47) Ted mapped  $E_2$  into  $E_2$  like this:

$$X_{\text{new}} = -X_{\text{old}}$$

$$Y_{\text{new}} = Y_{\text{old}}$$

Can you find the image of  $(0, 0)$ , using Ted's mapping? Can you find the image of  $(1, 0)$ ? Can you find the image of  $(0, 1)$ ? Can you show the mapping of  $\{(1, 0), (0, 0), (0, 1)\}$ , using Al's method?

(47) Again, let's just "play around" for a moment; let's try a few points and see what happens:

$$T: (0, 0) \longrightarrow (0, 0)$$

$$T: (1, 0) \longrightarrow (-1, 0)$$

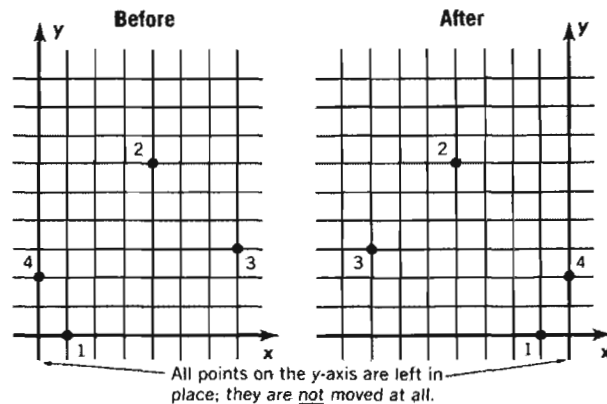
$$T: (0, 2) \longrightarrow (0, 2)$$

$$T: (4, 6) \longrightarrow (-4, 6)$$

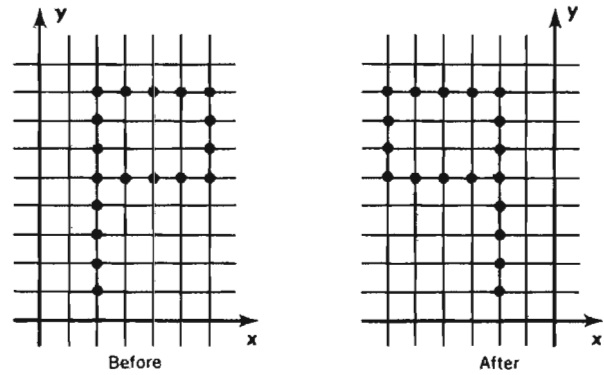
$$T: (7, 3) \longrightarrow (-7, 3)$$

(All points on the  $y$ -axis are left where they were! They are not moved at all!)

Let's plot a few of these:



... and the mapping  $T$  "reverses" everything symmetrically about the  $y$ -axis. Mathematicians call this a "reflection in the  $y$ -axis."



Using Al's method, the mapping of  $\{(1, 0), (0, 0), (0, 1)\}$  is

$$T: (1, 0) \rightarrow (-1, 0)$$

$$T: (0, 0) \rightarrow (0, 0)$$

$$T: (0, 1) \rightarrow (0, 1)$$

(48) Draw some figure in  $E_2$ . Can you find its image, using Ted's mapping?

(48) See the answer to question 47.

## CANDY-STORE ARITHMETIC

You may want to view the film entitled "Introduction to Matrix Multiplication." This topic appears also, briefly, in the films "Solving Equations with Matrices" and "Matrices."

For a good reference on matrix multiplication, see Kemeny (82).



### CHAPTER 37

## Candy-Store Arithmetic

[page 131]

Mathematicians have often built important and elaborate mathematical systems by starting with a very "commonplace" idea, which they have been able to extend in some significant way. Let's see if we can do it.

(1) Andy goes to a candy store that sells peppermints (for 2¢ each), chocolate almond bars (for 10¢ each), and chocolate-covered ants (for 50¢ a box). We can write this as:

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

Suppose that today Andy buys three peppermints, one chocolate almond bar, and zero boxes of chocolate-covered ants (as a matter of fact, Andy always buys zero boxes of chocolate-covered ants). We can write this as:

$$(3 \ 1 \ 0)$$

The numbers

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

are called a **price matrix**, and the numbers

$$(3 \ 1 \ 0)$$

are called a **demand matrix**.\*

Can you multiply the demand matrix

$$(3 \ 1 \ 0)$$

## ANSWERS AND COMMENTS

$$\begin{aligned} (1) \quad (3 \ 1 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (3 \times 2) + (1 \times 10) + (0 \times 50) \\ &= 6 + 10 + 0 \\ &= 16¢ \end{aligned}$$

\*With the introduction of matrices we are turning to the mathematics of quite recent times. Indeed, the algebra of matrices was introduced in the year 1857 by the English mathematician Arthur Cayley (1821-1895). (The singular form is *matrix*. The plural form of the word is *matrices*.)

by the price matrix

$$\begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix}$$

to get the amount of money that Andy spent?

$$(3 \ 1 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} = ?$$

(2) Joan says that you write:

$$\begin{aligned} (3 \ 1 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (3 \times 2) + (1 \times 10) + (0 \times 50) \\ &= 6 + 10 + 0 \\ &= 16\text{¢} \end{aligned}$$

Do you agree?

[page 132]

(3) Nancy says that you write:

$$\begin{aligned} (3 \ 1 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (3 \times 2) + (1 \times 10) + (0 \times 50) \\ &= 6 + 10 + 0 \\ &= 16\text{¢} \end{aligned}$$

Do you agree?

(4) Jill says that you write:

$$\begin{aligned} (3 \ 1 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (3 \times 50) + (1 \times 10) + (0 \times 2) \\ &= 150 + 10 + 0 \\ &= 160\text{¢} \end{aligned}$$

Do you agree?

(5) Suppose that Andy goes to the store on Thursday and buys

$$(4 \ 2 \ 0).$$

What did he buy? How much money did he spend?

(6) Toby went to the store and bought

$$(1 \ 3 \ 0).$$

What did he buy? How much money did he spend?

(2) **No.** Compare with the answer to question 1.

(3) **Yes.**

(4) **No.**

(5) **Andy bought 4 peppermints, 2 chocolate almond bars, 0 boxes of chocolate covered ants. He spent:**

$$\begin{aligned} (4 \ 2 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (4 \times 2) + (2 \times 10) + (0 \times 50) \\ &= 8 + 20 + 0 \\ &= 28\text{¢} \end{aligned}$$

(6) **Toby bought 1 peppermint, 3 chocolate almond bars, 0 boxes of chocolate covered ants. Toby spent:**

$$\begin{aligned} (1 \ 3 \ 0) \times \begin{pmatrix} 2 \\ 10 \\ 50 \end{pmatrix} &= (1 \times 2) + (3 \times 10) + (0 \times 50) \\ &= 2 + 30 + 0 \\ &= 32\text{¢} \end{aligned}$$

(7) One day the store had a special sale. For that day only their prices were

$$\begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix}$$

How much did each item cost at the sale price?

(8) On the day of the sale, Nancy bought

$$(4 \ 3 \ 0).$$

What did Nancy buy? How much money did she spend?

(9) Up until now we have been dealing with a candy store that sells peppermints and chocolate almond bars, and tries to sell boxes of chocolate-covered ants.

Suppose we now try to build an abstract system. We will forget all about stores and prices and quantities. All we will remember is the **pattern** of what we have been doing. Using this same pattern, let's multiply these two matrices:

$$(5 \ 2) \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} = ?$$

(10) Can you use this same pattern to multiply the following two matrices?

$$(1 \ 3 \ 0 \ 2) \times \begin{pmatrix} 5 \\ 0 \\ 19 \\ 8 \end{pmatrix} = ?$$

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(11) Suppose that someone knew all about  $\square$ ,  $\triangle$ ,  $a$ ,  $b$ ,  $c$ ,  $x$ ,  $y$ , ... notation, but did not know matrix notation. Could you write something that would show him immediately how matrix notation works?

(7) Peppermints were 1¢ each, chocolate almond bars were 5¢ each, and chocolate covered ants were 25¢ per box.

(8) Nancy bought 4 peppermints, 3 chocolate almond bars, 0 boxes of chocolate covered ants. She spent:

$$\begin{aligned} (4 \ 3 \ 0) \times \begin{pmatrix} 1 \\ 5 \\ 25 \end{pmatrix} &= (4 \times 1) + (3 \times 5) + (0 \times 25) \\ &= 4 + 15 + 0 \\ &= 19¢ \end{aligned}$$

(9)  $(5 \ 2) \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} = (5 \times 4) + (2 \times 3)$   
 $= 20 + 6$   
 $= 26$

(10)  $(1 \ 3 \ 0 \ 2) \times \begin{pmatrix} 5 \\ 0 \\ 19 \\ 8 \end{pmatrix} = (1 \times 5) + (3 \times 0) + (0 \times 19) + (2 \times 8)$   
 $= 5 + 0 + 0 + 16$   
 $= 21$

(11)  $(A \ B \ C) \times \begin{pmatrix} W \\ X \\ Y \end{pmatrix} = (A \times W) + (B \times X) + (C \times Y)$

This use of variables to enable us to "pass instructions along to our assistant" is of great importance. Just to be on the safe side, let's pause for a moment and make sure that we agree that this notation really does do what we claim it does.

Since we will want to indicate replacements of variables by numbers, it may be convenient to rewrite our answer to question 11, using frames instead of letters. (This makes no real difference; we do it only for convenience.) The answer to question 11 could, then, be written:

$$(\square \ \triangle \ \nabla) \times \begin{pmatrix} \text{parallelogram} \\ \text{trapezoid} \\ \text{trapezoid} \end{pmatrix} = (\square \times \text{parallelogram}) + (\triangle \times \text{trapezoid}) + (\nabla \times \text{trapezoid})$$

The rule for substituting now does the rest of the job! For example, whatever number is put in the first  $\square$  *must be put in all the*  $\square$ 's (there is one other). If, for example, this number were 3, we should have

$$\left( \begin{array}{ccc} \square & \triangle & \nabla \end{array} \right) \times \left( \begin{array}{c} \square \\ \square \\ \square \end{array} \right) = \left( \begin{array}{c} \square \\ \square \\ \square \end{array} \right) + \left( \begin{array}{c} \triangle \\ \triangle \\ \triangle \end{array} \right) + \left( \begin{array}{c} \nabla \\ \nabla \\ \nabla \end{array} \right)$$

If you continue in this way, you will see that, once we have written

$$\left( \begin{array}{ccc} \square & \triangle & \nabla \end{array} \right) \times \left( \begin{array}{c} \square \\ \square \\ \square \end{array} \right) = \left( \begin{array}{c} \square \\ \square \\ \square \end{array} \right) + \left( \begin{array}{c} \triangle \\ \triangle \\ \triangle \end{array} \right) + \left( \begin{array}{c} \nabla \\ \nabla \\ \nabla \end{array} \right),$$

the rule for substitution now *compels* us to use correctly what is called the "inner product" pattern!

(12) Jane says she could show him matrix notation by writing:

$$\left( \begin{array}{ccc} A & B & C \end{array} \right) \times \left( \begin{array}{c} D \\ E \\ F \end{array} \right) = (A + D) \times (B + E) \times (C + F)$$

Do you agree?

(12) **Jane has, of course, written the pattern incorrectly. Her  $\times$  and + signs are located incorrectly.**

The correct answer would be

$$\left( \begin{array}{ccc} A & B & C \end{array} \right) \times \left( \begin{array}{c} D \\ E \\ F \end{array} \right) = (A \times D) + (B \times E) + (C \times F),$$

which is identical with our answer to question 11. (It makes no difference whether you mark a location in the formula with A or  $\square$  or whatever, just so long as the *same* symbol is used

$$\begin{array}{c} (\alpha \quad \beta \quad \gamma) \times \left( \begin{array}{c} \sigma \\ \epsilon \\ \lambda \end{array} \right) = (\alpha \times \sigma) + (\beta \times \epsilon) + (\gamma \times \lambda) \\ \uparrow \qquad \qquad \qquad \uparrow \\ \text{here} \qquad \qquad \qquad \text{and also here} \end{array}$$

and *nowhere else!* – and so on for all the other variables.)

(13) Hal says he could explain matrix notation by writing:

$$\left( \begin{array}{ccc} A & B & C \end{array} \right) \times \left( \begin{array}{c} A \\ B \\ C \end{array} \right) = (A \times A) + (B \times B) + (C \times C)$$

Do you agree?

(13) **No. Hal's paper will only allow us to multiply when the first two numbers are the same, the second two numbers are the same, etc. But it is not enough to be able to multiply**

$$\left( \begin{array}{ccc} 3 & 4 & 0 \end{array} \right) \times \left( \begin{array}{c} 3 \\ 4 \\ 0 \end{array} \right)$$

**We also want to be able to multiply**

$$\left( \begin{array}{ccc} 3 & 4 & 0 \end{array} \right) \times \left( \begin{array}{c} 5 \\ 6 \\ 3 \end{array} \right),$$

**and so on.**

(14) Ellen says she could explain matrix notation by writing:

$$(A \ B \ C) \times \begin{pmatrix} D \\ E \\ F \end{pmatrix} = (A \times D) + (B \times E) + (C \times F)$$

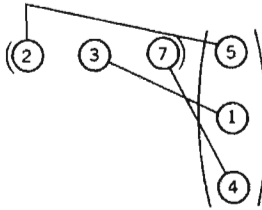
What do you think?

(15) Courtney says he would write:

$$\begin{pmatrix} \square & \triangle & \nabla \end{pmatrix} \times \begin{pmatrix} \text{parallelogram} \\ \text{rectangle} \\ \text{trapezoid} \end{pmatrix} = (\square \times \text{parallelogram}) + (\triangle \times \text{rectangle}) + (\nabla \times \text{trapezoid})$$

What do you think?

(16) Joe says that the way to multiply matrices is by pairs

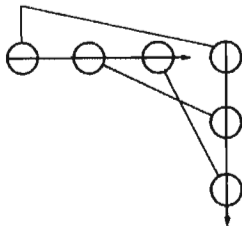


and "from left to right on the left and from top to bottom on the right,"



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which we might also draw like this:



Do you think this is a good description?

(17) Can you multiply these matrices?

$$(3 \ 7 \ 10) \times \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = ?$$

(14) Ellen is right.

(15) Courtney is also correct.

(16) I do—but every man to his own taste.

$$\begin{aligned} (17) (3 \ 7 \ 10) \times \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} &= (3 \times 2) + (7 \times 5) + (10 \times 8) \\ &= 6 + 35 + 80 \\ &= 121 \end{aligned}$$

(18) Can you multiply these matrices?

$$\begin{pmatrix} 8 & 12 & 15 \end{pmatrix} \times \begin{pmatrix} 10 \\ 5 \\ 2 \end{pmatrix} = ?$$

(19) Can you multiply these matrices?

$$\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} W \\ X \\ Y \end{pmatrix} = ?$$

(20) Can you extend the idea of matrix multiplication, still using the same pattern, so that you can multiply these matrices?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = ?$$

$$\begin{aligned} (18) \quad \begin{pmatrix} 8 & 12 & 15 \end{pmatrix} \times \begin{pmatrix} 10 \\ 5 \\ 2 \end{pmatrix} &= (8 \times 10) + (12 \times 5) + (15 \times 2) \\ &= 80 + 60 + 30 \\ &= 170 \end{aligned}$$

(19) See the answer to question 11.

(20) Here we come to something new! Since we are asked to "extend" something, we are entitled to be creative. Perhaps no answer is *absolutely* right, or *absolutely* wrong. You may want to entertain any reasonable suggestion from any of your students. However, you also want to end up with the same system that professional mathematicians have agreed to use. Here is how it goes:

(i) The "answer" is itself a matrix, with two numbers in it:

$$\begin{pmatrix} \square \\ \triangle \end{pmatrix} \leftarrow \begin{array}{l} \text{One number here} \\ \text{The other number here} \end{array}$$

(ii) We always take our *rows* from the left-hand matrix and our *columns* from the right-hand matrix. Hence, we continue to use this pattern:



(iii) For the *first row, first column* number in the answer, we choose the *first row* (from the left-hand matrix) and the *first column* (from the right-hand matrix):

$$\text{Use this row} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \square \\ \triangle \end{pmatrix} \leftarrow$$

$$(1 \times 5) + (2 \times 7) + (3 \times 4)$$

$$5 + 14 + 12$$

$$31 \rightarrow \square$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 31 \\ \triangle \end{pmatrix}$$

(iv) For the *second row, first column* number in the answer, we choose the *second row* (from the left-hand matrix) and the *first column* (from the right-hand matrix):



Use this row  $\rightarrow$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 31 \\ \triangle \end{pmatrix}$$

$$(0 \times 5) + (11 \times 7) + (10 \times 4)$$

$$0 + 77 + 40$$

$$117 \rightarrow \triangle$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 31 \\ 117 \end{pmatrix}$$

(v) The matrix

$$\begin{pmatrix} 31 \\ 117 \end{pmatrix}$$

is the final answer. We do not try to "simplify" it any further.

(21) Lex says the idea is to use rows from the left-hand matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix}$$

and to use columns from the right-hand matrix.

$$\begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix}$$

What do you think?

(22) Ellen says that the answer has this form:  
A matrix with some number here  $\rightarrow$ 

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \square \\ \square \end{pmatrix}$$

and with some number here  $\rightarrow$ 

If we write

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} \square \\ \triangle \end{pmatrix}$$

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what number should go in the  $\square$  in order to make a true statement?(23) What number should go in the  $\triangle$ ?

(23) 117 (See the answer to question 20.)

(24) To find the  $\square$  number, Eva wrote this:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 11 & 10 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix}$$

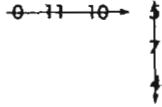
$$(1 \times 5) + (2 \times 7) + (3 \times 4) = 5 + 14 + 12$$

$$= 31$$

Do you agree?

(24) Yes

(25) To find the  $\triangle$  number, Marilyn wrote this:



$$(0 \times 5) + (11 \times 7) + (10 \times 4) = 77 + 40 = 117$$

Do you agree?

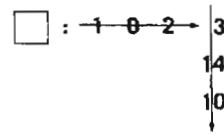
(26) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 0 & 11 \end{pmatrix} \times \begin{pmatrix} 3 \\ 14 \\ 10 \end{pmatrix} = \begin{pmatrix} \square \\ \triangle \end{pmatrix}$$

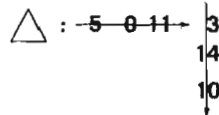
(Nancy says the answer should be a matrix with two numbers in it. Do you agree?)

(25) Yes

$$(26) \begin{pmatrix} 1 & 0 & 2 \\ 5 & 0 & 11 \end{pmatrix} \times \begin{pmatrix} 3 \\ 14 \\ 10 \end{pmatrix} = \begin{pmatrix} \square \\ \triangle \end{pmatrix}$$



$$(1 \times 3) + (0 \times 14) + (2 \times 10) = 3 + 0 + 20 = 23 \rightarrow \square$$



$$(5 \times 3) + (0 \times 14) + (11 \times 10) = 15 + 0 + 110 = 125 \rightarrow \triangle$$

Thus the final answer is the matrix

$$\begin{pmatrix} 23 \\ 125 \end{pmatrix}$$

that is,

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 0 & 11 \end{pmatrix} \times \begin{pmatrix} 3 \\ 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 23 \\ 125 \end{pmatrix}$$

(27) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 5 & 7 \\ 10 & 3 & 0 \\ 4 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ 11 \end{pmatrix} = \begin{pmatrix} \square \\ \triangle \\ \nabla \end{pmatrix}$$

(27) Here, again, we must extend a bit further; but by now the matter may be clear. You have three rows (in the left-hand matrix) and one column (in the right-hand matrix). Consequently, the answer will be a matrix with three rows and one column!

$$\begin{pmatrix} \square \\ \triangle \\ \nabla \end{pmatrix}$$

$$\square : (1 \times 2) + (5 \times 9) + (7 \times 11) = 2 + 45 + 77 = 124 \rightarrow \square$$

$$\begin{array}{r} \triangle : (10 \times 2) + (3 \times 9) + (0 \times 11) \\ \quad 20 \quad + \quad 27 \quad + \quad 0 \\ \quad \quad \quad 47 \rightarrow \triangle \end{array}$$

$$\begin{array}{r} \nabla : (4 \times 2) + (6 \times 9) + (8 \times 11) \\ \quad 8 \quad + \quad 54 \quad + \quad 88 \\ \quad \quad \quad 150 \rightarrow \nabla \end{array}$$

Thus we have

$$\begin{pmatrix} 1 & 5 & 7 \\ 10 & 3 & 0 \\ 4 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ 11 \end{pmatrix} = \begin{pmatrix} 124 \\ 47 \\ 150 \end{pmatrix}.$$

(28) Hal says the answer should be a matrix like this:

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

Do you agree?

(29) Tom says the answer to question 27 should be a matrix like this:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Do you agree?

(30) Jane says the answer to question 27 should be a matrix like this:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

What do you think?

(31) Can you extend the idea of matrix multiplication, so that you can multiply "2 - by - 2" matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ G & H \end{pmatrix} = \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix}$$

(28) **No.** See the answer to question 27.

(29) **Yes**

(30) **No**

(31) **Here we extend one more time! But the pattern by now may be clear.**

(i) **As always, take rows from the left-hand matrix and columns from the right-hand matrix.**

(ii) **For, say, the second row, first column number in the answer**

$$\begin{pmatrix} \text{---} & \text{---} \\ \square & \text{---} \end{pmatrix},$$

**select the second row (from the left-hand matrix) and the first column (from the right-hand matrix).**

$$\begin{pmatrix} A & B \\ \leftarrow D \end{pmatrix} \times \begin{pmatrix} W & X \\ G \downarrow & H \end{pmatrix} = \begin{pmatrix} \text{---} & \text{---} \\ \square & \text{---} \end{pmatrix}$$

(iii) **Once you have selected the correct row and column, you proceed as in questions 1, 3, 5, 6, and 8 (see also questions 9 and 10).**

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(32) Lex says that the idea is to use rows from the left-hand factor

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow$$

and columns from the right-hand factor:

$$\begin{pmatrix} W & X \\ G & H \end{pmatrix} \downarrow$$

What do you think?

(33) Ellen says that

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & 7 \\ 6 & 11 \end{pmatrix} = \begin{pmatrix} 42 & 76 \\ 14 & 25 \end{pmatrix}$$

What do you think?

(34) Jerry says that Ellen got the "42" by saying

$$(3 \times 4) + (5 \times 6) = 42.$$

What do you think?

(35) How did Ellen get the "25"?

(36) How did Ellen get the "76"?

Here is the result:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ G & H \end{pmatrix} = \begin{pmatrix} (A \cdot W) + (B \cdot G) & (A \cdot X) + (B \cdot H) \\ (C \cdot W) + (D \cdot G) & (C \cdot X) + (D \cdot H) \end{pmatrix}$$

(32) Lex is right.

(33) Ellen is correct (See the answers to questions 34 through 37).

(34) That is correct. (This was the *first row, first column* number in the answer, so Ellen chose first row and first column, following Lex's advice in question 32.)

(35) This is the *second row, second column* number in the answer. Hence (remember question 32), we choose the second row (from the left-hand matrix)

$$2 \quad 1$$

and the second column (from the right-hand matrix)

$$7$$

$$11,$$

and then proceed as in question 9:

$$(2 \times 7) + (1 \times 11)$$

$$14 \quad + \quad 11$$

$$25$$

So Ellen was right on this number.

(36) The "76" is the *first row, second column* number in the answer. Hence, we select the first row (in the left-hand matrix) and the second column (in the right-hand matrix):

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & 7 \\ 6 & 11 \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

$$\begin{array}{r} (3 \times 7) + (5 \times 11) \\ 21 + 55 \\ 76 \rightarrow \square \end{array}$$

Ellen was again right.

(37) How did Ellen get the "14"?

(37) The "14" is the *second row, first column* number in the answer

$$\begin{array}{c} \text{First column} \\ \downarrow \\ \text{Second row} \rightarrow \left( \begin{array}{cc} \square & \square \end{array} \right) \end{array}$$

so we select the *second row* (from the left-hand matrix) and the *first column* (from the right-hand matrix):

$$\left( \begin{array}{cc} 3 & 5 \\ 2 & 1 \end{array} \right) \times \left( \begin{array}{c} 4 \\ 6 \\ 11 \end{array} \right) = \left( \begin{array}{c} \square \\ \square \end{array} \right)$$

$$\begin{array}{r} (2 \times 4) + (1 \times 6) \\ 8 + 6 \\ 14 \rightarrow \square \end{array}$$

Ellen, once again, was right!

(38) Can you multiply these two matrices?

$$\left( \begin{array}{cc} 7 & 1 \\ 0 & 2 \end{array} \right) \times \left( \begin{array}{cc} 3 & 5 \\ 1 & 8 \end{array} \right) = \left( \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right)$$

$$(38) \left( \begin{array}{cc} 7 & 1 \\ 0 & 2 \end{array} \right) \times \left( \begin{array}{cc} 3 & 5 \\ 1 & 8 \end{array} \right) = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$$

$$\begin{aligned} A &= (7 \times 3) + (1 \times 1) \\ &= 21 + 1 = 22 \end{aligned}$$

$$\begin{aligned} B &= (7 \times 5) + (1 \times 8) \\ &= 35 + 8 \\ &= 43 \end{aligned}$$

$$\begin{aligned} C &= (0 \times 3) + (2 \times 1) \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} D &= (0 \times 5) + (2 \times 8) \\ &= 0 + 16 \\ &= 16 \end{aligned}$$

Thus we have

$$\left( \begin{array}{cc} 7 & 1 \\ 0 & 2 \end{array} \right) \times \left( \begin{array}{cc} 3 & 5 \\ 1 & 8 \end{array} \right) = \left( \begin{array}{cc} 22 & 43 \\ 2 & 16 \end{array} \right)$$

(39) Amy knows all about  $\square, \triangle, A, B, C, \dots$ , and so on, but she does not know how to multiply 2-by-2 matrices. Can you write something that will show Amy how to multiply 2-by-2 matrices?

$$(39) \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \times \left( \begin{array}{c} \square \\ \triangle \\ \nabla \\ \square \end{array} \right) =$$

$$\left( \begin{array}{c} (A \times \square) + (B \times \nabla) \\ (C \times \square) + (D \times \nabla) \end{array} \right) \left( \begin{array}{c} (A \times \triangle) + (B \times \square) \\ (C \times \triangle) + (D \times \square) \end{array} \right)$$

(40) Jerry wrote:

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 12 & 7 \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

(41) Steve wrote:

$$\begin{pmatrix} \square & \triangle \\ \nabla & \square \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \\ = \begin{pmatrix} (\square \times W) + (\triangle \times Y) & (\square \times X) + (\triangle \times Z) \\ (\nabla \times W) + (\square \times Y) & (\nabla \times X) + (\square \times Z) \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

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(42) Mary wrote:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \\ = \begin{pmatrix} (A \times W) + (B \times Y) & (A \times X) + (B \times Z) \\ (C \times W) + (D \times Y) & (C \times X) + (D \times Z) \end{pmatrix}$$

Does this show Amy how to multiply 2-by-2 matrices?

(43) Who is right, Steve or Mary?

Can you multiply these 2-by-2 matrices?

$$(44) \begin{pmatrix} 1 & 7 \\ 2 & 0 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$(45) \begin{pmatrix} 5 & 1 \\ 3 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$(46) \begin{pmatrix} 5 & 7 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$(47) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(48) Jill says that we add matrices like this:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} A + W & B + X \\ C + Y & D + Z \end{pmatrix}$$

What do you think?

(49) Can you add these two matrices?

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 3 & 10 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(40) **No; you cannot retrace Jerry's steps merely by looking at his answer.**(41) **Yes. Now all Amy needs to do is to use UV.**(42) **Yes. Again, all Amy needs to do is to use UV.**(43) **Both. They both say the same thing.**

$$(44) \begin{pmatrix} 38 & 4 \\ 6 & 8 \end{pmatrix}$$

$$(45) \begin{pmatrix} 10 & 1 \\ 6 & 0 \end{pmatrix}$$

$$(46) \begin{pmatrix} 10 & 14 \\ 6 & 4 \end{pmatrix}$$

$$(47) \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

(48) **Jill is correct; this is the way that mathematicians have agreed to add matrices.**

This is, of course, a rhetorical question. There is no way (ordinarily, at this point) for the students to know. Nonetheless, we prefer to ask this question anyway—perhaps because it helps to get the children's attention.

$$(49) \begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 3 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 10 & 15 \end{pmatrix}$$

(50) Can you write some axioms for arithmetic and algebra?

(51) Do you know an identity that is called the "commutative law for addition"?

(52) In arithmetic and algebra, what is special about the number 1?

(53) Can you write the identity that is known as the "law for 1"?

(54) In arithmetic and algebra, what is special about the number 0?

(55) Can you write the identity that is known as the "addition law for zero"?

(56) Can you make up an "addition law for zero" that will apply to 2-by-2 matrices?

(57) What 2-by-2 matrix corresponds to the number 0?

(58) Is there a "multiplication law for zero" that works for matrices?

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(59) Can you write what might be called a "law for 1" for matrices, instead of numbers?

(50) Hopefully, the students can write a list of such axioms  $\square = \square$ ,  $\square + \triangle = \triangle + \square$ ,  $\square \times 1 = \square$ , and so forth).

$$(51) \square + \triangle = \triangle + \square$$

(52)  $\square \times 1 = \square$  is an identity (that is, multiplying by 1 "doesn't change anything").

$$(53) \square \times 1 = \square$$

(54)  $\square + 0 = \square$ , (that is, adding zero "doesn't change anything").

$$(55) \square + 0 = \square$$

$$(56) \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$(57) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(58) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(59) Here we reach a crisis! The obvious thing to try is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

that is,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

... but when we multiply these last two matrices, we encounter trouble...

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} A + B & A + B \\ C + D & C + D \end{pmatrix}$$

and

$$\begin{pmatrix} A + B & A + B \\ C + D & C + D \end{pmatrix} \neq \begin{pmatrix} A & B \\ C & D \end{pmatrix}!$$

Here we face a dilemma: Our attempt to find a matrix that "behaves like the number 1" has failed. Is it because there is no matrix that behaves like the number 1? Or is it because we have, ourselves, made some error or left our work unfinished? In the first case, any further search will probably be in vain. In the second case, all we need to do is to get back on the job; persistence and shrewdness will ultimately be rewarded.

But which is it?

This point is made very clearly in the filmed lesson entitled "Matrices"; it would be worth your while to view this film at this point.

In point of fact, there is a matrix that "behaves like the number 1." It is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and, using it, we get a perfect parallel:

Numbers

$$\square \times 1 = \square$$

2-by-2 Matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(60) Jerry says the "law for 1" for matrices would look like this:

$$\begin{pmatrix} \text{Any 2-by-2} \\ \text{matrix} \end{pmatrix} \times \begin{pmatrix} \uparrow \\ \phantom{\uparrow} \end{pmatrix} = \begin{pmatrix} \text{The same 2-by-2} \\ \text{matrix that you} \\ \text{started with} \end{pmatrix}$$

The "1" matrix, whatever that may be

Can you use the  $A, B, C, D, \dots$  notation to write Jerry's law?

(61) Mary wrote this:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} \uparrow \\ \phantom{\uparrow} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The "1" matrix, if there really is any

What do you think?

(62) Can you find the "1" matrix to put into "Mary's law"?

(63) Don says the "1" matrix should be

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(64) Can you write the "law for 1" for matrices? What is the "1" matrix?

$$(60) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} \phantom{1} & \phantom{0} \\ \phantom{0} & \phantom{1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

↑  
The "1" matrix, if there is one (you and I know there is, but the students may not at this stage)

(61) Mary is correct.

(62) See the answer to question 59.

(63) This is a good guess; unfortunately, it won't work. (See the answer to question 59.)

(64) See the answer to question 59.

Notice that this is another typical Madison Project sequence of questions. Question 59 states the task (a student *might* be able to answer it completely at this point). Questions 60 through 63 contain hints, methods of attack, and so on — they "nibble away at the problem," breaking it into a sequence of easier questions. Finally, question 64 is a restatement of question 59; by this point the students should be able to answer it.



(65) George says that, for numbers, there is an axiom that says that every number except zero has a multiplicative inverse, so that

$$A \times \bar{A} = 1.$$

Do you think this axiom applies to the system of matrices?

(65) Well, more or less. The parallel is not perfect:

The number 0 has no multiplicative inverse.

The 2-by-2 matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has no multiplicative inverse.

The numbers 2, 3, 4, etc., have the inverses  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ :  $2 \times \frac{1}{2} = 1$ ,  $3 \times \frac{1}{3} = 1$ , and so on.

Some 2-by-2 matrices have multiplicative inverses:

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Some 2-by-2 matrices are *not* the zero matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and yet they have no inverses; here is one:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Here is another:

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}$$

There is no parallel, among numbers, for the matrices that are *not* zero, but *still don't have inverses*.

(However, after you finish Chapter 47 you may want to return to this problem and think about it some more.)

(66) Can you multiply these two matrices?

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(67) George says that if we use  $A$  to mean

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix},$$

then the multiplicative inverse  $\bar{A}$  would be

$$\begin{pmatrix} \frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}.$$

What do you think?

$$(66) \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} \frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

is the *multiplicative inverse* of

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}.$$

(67) George is right.

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(68) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$$

(69) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{3}{7} \end{pmatrix}$$

(70) Do the matrices

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{3}{7} \end{pmatrix}$$

satisfy the commutative law for multiplication?

(71) Do all matrices satisfy the commutative law for multiplication?

(72) Can you multiply these two matrices?

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$(68) \begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{3}{7} \end{pmatrix}$$

See the answer to question 66.

(69) If you haven't already noticed it, you should now notice that matrix multiplication is *not always commutative*. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix},$$

whereas

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix},$$

so that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, if we use 2-by-2 matrices as replacements for the variables  $\square$  and  $\triangle$ , then  $\square \times \triangle = \triangle \times \square$  is *not an identity!*

Now that we have noticed that matrix multiplication does *not always commute*, we can be properly surprised to find that every matrix commutes with its own inverse!

Hence,

$$\begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{3}{7} \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}$$

is the multiplicative inverse of

$$\begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{3}{7} \end{pmatrix}.$$

(70) Yes, *they do*; but this is *not* typical of matrix multiplication in general. (See the answer to question 69.)

(71) No. Sometimes a particular pair of matrices will commute, and a few matrices commute with any other matrix, but these cases are exceptional. In general, matrices do *not* satisfy CLM.

$$(72) \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

(73) Can you multiply these two matrices?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(74) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

(75) Can you find the multiplicative inverse of this matrix?

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Can you find the multiplicative inverses for these matrices?

(76)  $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(77)  $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$  where  $p$  is a number, and  $p \neq 0$ .

(78)  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(79)  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(80)  $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

(81)  $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(82)  $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(83)  $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(84)  $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

(85)  $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$

(73)  $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$

(Compare the answers to questions 71, 72, and 73.)

(74)  $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$  since  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(75)  $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$  since  $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(76)  $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

(77)  $\begin{pmatrix} {}^r p & 0 \\ 0 & {}^r p \end{pmatrix}$ , where  ${}^r p$  denotes the multiplicative inverse of the number  $p$ .

(78)  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This matrix is its own inverse!

(79)  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This matrix is its own inverse!

(80)  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(81)  $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

(82)  $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$

(83)  $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

(84)  $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

(85)  $\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}$

(86)  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(86) We can easily *prove* that this matrix *has no inverse*:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, no matter what matrix we put here, the answer will always be

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence will *never* be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has no inverse.

(87)  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(87) Again, we can *prove* that this matrix has no inverse:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

No matter what 2-by-2 matrix we write here, the "answer" matrix will always have a zero in the lower right-hand corner:

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

Therefore it will never be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that we *can* use UV:  $1 \rightarrow A, 0 \rightarrow B$ ; but this only gives us the first row; there is no way to get the 1 in the lower right-hand corner.)

(88)  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(88) Again,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + C & B + D \\ 0 & 0 \end{pmatrix}.$$

Hence there can be no inverse. (See the answer to question 87.)

(89) Can you add these two matrices?

$$\begin{pmatrix} 1 & 9 \\ 11 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

(89)  $\begin{pmatrix} 4 & 14 \\ 17 & 6 \end{pmatrix}$

(90) Can you find an additive inverse for the number  $^{-}7$ ?(90)  $^{-}7$ ; that is,  $^{-}7 + ^{-}7 = 0$ .

(91) Can you find an additive inverse for the number  $-3$ ?

(91) **The additive inverse of  $-3$  is  $+3$ ; that is,  $-3 + +3 = 0$ .**

(92) Can you find an additive inverse for the number  $0$ ?

(92) **The additive inverse of  $0$  is  $0$ ; that is,  $0 + 0 = 0$ .**

Remember, when we say we are seeking the "additive inverse of the number  $p$ " what we mean is that we are concerned with the truth set for the open sentence  $p + \square = 0$ .

(93) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} +2 & +3 \\ 0 & +1 \end{pmatrix}$$

(93)  $\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$

That is,

$$\begin{pmatrix} +2 & +3 \\ 0 & +1 \end{pmatrix} + \begin{pmatrix} -2 & -3 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(94) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} -5 & +2 \\ -1 & -7 \end{pmatrix}$$

(94)  $\begin{pmatrix} +5 & -2 \\ +1 & -7 \end{pmatrix}$

(95) Can you find an additive inverse for every matrix?

(95) **Yes. This is exactly analogous to the situation with numbers.**

(96) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} +7 & +3 \\ -1 & 0 \end{pmatrix}$$

(96)  $\begin{pmatrix} -7 & -3 \\ +1 & 0 \end{pmatrix}$

(97) Can you find an additive inverse for this matrix?

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(97) **Yes, the additive inverse of**

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

that is,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

## RICKY'S SPECIAL MATRIX

This chapter continues (and in part retraces) the work of Chapter 37. The present chapter parallels very closely the content of the film "Matrices."

Matrices is a very open-ended subject, as we shall show even more forcefully in the following few chapters. One can continue finding fascinating new properties of matrices, and new uses for them, virtually indefinitely. There is no point of completion—no ultimate "finished" picture of the entire matrix story.

*Nowhere in the Madison Project materials is the "light touch" more important than in these lessons with matrices. We recommend that you drop the subject of matrices the moment they cease to fascinate you and your students—if possible, before they cease to fascinate you and your students.*

In an attempt to provide for various student responses, we have included far more material on matrices than you will probably wish to use. If our extensive treatment seems somewhat heavy-handed, we apologize, and suggest you follow our advice and not our example.



### CHAPTER 38

## Ricky's Special Matrix

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- (1) Can you multiply these two matrices?

$$\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 10 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (2) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (3) Ricky says that he has found a special matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Can you tell what is "special" about Ricky's matrix?

## ANSWERS AND COMMENTS

(1)  $\begin{pmatrix} 4 & 40 \\ 12 & 42 \end{pmatrix}$

(2)  $\begin{pmatrix} AR + BT & AS + BU \\ CR + DT & CS + DU \end{pmatrix}$

or, in full notation (which often seems preferable for use with children at this stage),

$$\begin{pmatrix} (A \times R) + (B \times T) & (A \times S) + (B \times U) \\ (C \times R) + (D \times T) & (C \times S) + (D \times U) \end{pmatrix}$$

- (3) It plays (for 2-by-2 matrices) the same role that the number 1 plays for numbers. Specifically, for numbers,

$$\square \times 1 = \square,$$

whereas, for 2-by-2 matrices, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(4) Can you find any other "special" matrices? What is "special" about them?

(4) There is no end to the list of possible "special" or "interesting" matrices. Here are a few (you probably should try writing some of this out for yourself, on scratch paper, as you read this list):

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  plays a role analogous to the number 2.

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$  plays a role analogous to the number 3.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  will reverse columns—for example,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ D & C \end{pmatrix};$$

also, it will reverse rows, if used as a *left* multiplier:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix}.$$

$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$  combines the properties of

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

that is, it reverses rows or columns, and *also doubles them*.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  reproduces the first column, but inserts zeros for the second column:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix};$$

also, if used as a *left* multiplier, it deals similarly with rows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$  combines properties of  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and of  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ .

Do you see what it does?

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ , when used as a *right* multiplier, leaves the first column unchanged, but doubles the second column:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} A & (2 \times B) \\ C & (2 \times D) \end{pmatrix};$$

also, when used as a *left* multiplier, it deals similarly with rows.

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , when used as a *right* multiplier, replaces each entry with the *sum* of all entries in its row:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} A + B & A + B \\ C + D & C + D \end{pmatrix}.$$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  replaces a matrix by its additive inverse (opposite); it plays a role analogous to the number -1.

Evidently, one could go on like this forever. Among other possibilities:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \dots$$

These possibilities are listed here only to convince you that the subject is, indeed, open-ended, and to help you be prepared for some of the matrices that your students may "discover."

Your students may not discover any of the matrices on this list, or they may discover other "interesting" matrices that are not on this list.

It is probably wise *not* to "show" them or "tell" them these matrices. Which "special" matrices—if any—they discover is relatively unimportant. What is important is that the children understand *what it is that their "special" matrix does that makes it "special."*

- (5) Mary says that she has found a special matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Can you tell what is "special" about Mary's matrix?

- (6) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (7) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (8) Jeff says that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a "special" matrix. What does it do?

- (5) **Mary's matrix, when used as a right multiplier, reverses columns:**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ D & C \end{pmatrix};$$

when used as a left multiplier, it reverses rows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix}.$$

(If you use Mary's matrix twice—on the same side—you get back to where you started.)

- (6)  $\begin{pmatrix} B & A \\ D & C \end{pmatrix}$

- (7)  $\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$ ; what is "special" about this?

- (8) **When used as a right multiplier, Jeff's matrix replaces the first column with zeros, but leaves the second column unchanged:**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix};$$

when used as a left multiplier, it replaces the first row with zeros, leaving the second row unchanged:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$



- (9) Can you multiply these two matrices?

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (10) What "special" thing did the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

do to the matrix

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix}?$$

Would it do the same thing to every 2-by-2 matrix?

Can you prove it?

- (11) Can you multiply these two matrices?

$$\begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (12) What "special" thing did

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ do to } \begin{pmatrix} 7 & 11 \\ 13 & 19 \end{pmatrix}?$$

[page 142]

Would it do the same thing to every 2-by-2 matrix?

Can you prove it?

- (13) Nora says that

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is a "special" matrix. She says she found out by multiplying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}.$$

What "special" thing does

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

do to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}?$$

Would it do the same thing to every 2-by-2 matrix?

- (14) Can you find any other "special" matrices?

- (15) Can you find a matrix that would just double each element of any 2-by-2 matrix?

- (16) Dexter says that "double each element" means: You start with any 2-by-2 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and multiply by your "special" matrix to get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} 2A & 2B \\ 2C & 2D \end{pmatrix}.$$

↑  
Dexter's special matrix

What do you think?

$$(9) \begin{pmatrix} 0 & 11 \\ 0 & 19 \end{pmatrix}$$

- (10) See the answer to question 8. The secret of
- proving*
- it, of course, is to use
- variables*
- rather than numbers.

$$(11) \begin{pmatrix} 11 & 7 \\ 19 & 13 \end{pmatrix}$$

- (12) It reversed the columns. The "proof" again depends upon using
- variables*
- :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ D & C \end{pmatrix}.$$

$$(13) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & A \\ C & C \end{pmatrix}$$

It leaves the first column unchanged, and replaces the second column with the first. (This may be a case where the algebraic equation expresses the idea more simply than words.) Since we have "proved" this by using variables ("A," "B," "C," and "D") instead of numbers, we know that it *would* do the same thing to every 2-by-2 matrix.

- (14) Again, there is no end to the potential supply. Drop the subject before it becomes tiresome.

$$(15) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

- (16) Dexter is right. You may want to emphasize to the children that "2A" was used here to mean "2 × A," "2B" means "2 × B", and so on.

- (17) Can you multiply these two matrices?

$$\begin{pmatrix} 3 & 5 \\ 7 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (18) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (19) Can you multiply these two matrices?

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- (20) Can you multiply these two matrices?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

[page 143]

- (21) What does the matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

do to the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

- (22) Can you find a "special" matrix that will turn

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

into

$$\begin{pmatrix} C & D \\ A & B \end{pmatrix} ?$$

What is the special trick in doing this?

$$(17) \begin{pmatrix} 9 & 15 \\ 21 & 3 \end{pmatrix}$$

$$(18) \begin{pmatrix} 4 \times A & 4 \times B \\ 4 \times C & 4 \times D \end{pmatrix}$$

- (19) The result is the same as in question 18; the matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

has the peculiar property that it commutes with every 2-by-2 matrix:

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

An interesting (and pleasantly easy) question is: find the set of all 2-by-2 matrices with this same property (namely, that they commute with every 2-by-2 matrix.)

$$(20) \begin{pmatrix} 2 \times B & 2 \times A \\ 2 \times D & 2 \times C \end{pmatrix}$$

- (21) According to what happened in question 20, it reversed the columns, and doubled every entry.

What would have happened if we had used

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

as a left multiplier, instead:

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = ?$$

- (22) The "special trick" is that we must use our "special" matrix on the left, whereas in questions 1 through 21 we have been writing our "special" matrices on the right.

Here is the solution to question 22:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix}.$$

## MATRICES: A NEW MATHEMATICAL SYSTEM



### CHAPTER 39

## Matrices: A New Mathematical System

[page 144]

We have now built up a new mathematical system – namely, the system of 2-by-2 matrices. Although, like much modern mathematics, this system was made up “just for fun,” it turns out to be a very valuable system. If you continue the study of mathematics, you will find yourself using this system again and again.

But this is not our concern right now.

We have created a new mathematical system. Let us now explore it! See if you can think of any interesting questions to ask.

Here are some that other people have asked.

(1) Jean wants to know: Do 2-by-2 matrices satisfy the commutative law for addition?

(2) Hal wants to know: Do 2-by-2 matrices satisfy the commutative law for multiplication?

(3) Jerry wants to know: Do 2-by-2 matrices satisfy the addition law for zero?

(4) Ellen wants to know: Do 2-by-2 matrices satisfy the multiplication law for zero?

(5) Andy says that every integer or rational number has an additive inverse. Does every 2-by-2 matrix have an additive inverse?

## ANSWERS AND COMMENTS

(1) **Yes. To prove it, you should use variables.**

(2) **No, not in general. For example,**

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 1 & 2 \end{pmatrix}$$

so that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Compare the answers to question 69, Chapter 37, and question 19, Chapter 38.

(3) **Yes. See Chapter 37.**

(4) **Yes. See Chapter 37.**

(5) **Yes. See Chapter 37.**

(6) Does every number have a multiplicative inverse?

(6) Yes, except for zero.

(7) Does every 2-by-2 matrix have a multiplicative inverse?

(7) No. See the answer to question 65, Chapter 37.

(8) Do you know what mathematicians mean by algebraic closure?

(8) A set  $S$  is *algebraically closed* under a binary operation, denoted by  $\star$ , if\*

$$[(\alpha \in S) \text{ and } (\beta \in S)] \Rightarrow (\alpha \star \beta) \in S.$$

Let's look at some examples:

(i) The positive integers are closed under addition because whenever you try to add two positive integers, you can always express the answer as a positive integer.

(ii) The positive integers are *not* closed under subtraction because if you try to subtract one positive integer from another, you may not be able to find a positive integer for an answer (because the answer may be a negative integer, as in  $10 - 13 = -3$ ).

(9) Is the set of positive integers closed under addition?

(9) Yes; see the answer to question 8.

(10) Is the set of positive integers closed under subtraction?

(10) No; see the answer to question 8.

(11) Is the set of positive integers closed under multiplication?

(11) Yes; that is to say, if you multiply a positive integer by a positive integer, the result will necessarily be a positive integer.

(12) Is the set of positive integers closed under division?

(12) No; for example,  $8 \div 3$  is not the name of a positive integer.

(13) Is the set of 2-by-2 matrices closed under addition?

(13) Yes; that is to say, if you add a 2-by-2 matrix to a 2-by-2 matrix, the result will be a 2-by-2 matrix.

(14) Is the set of 2-by-2 matrices closed under multiplication?

(14) Yes

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(15) Is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

closed under addition?

(15) Yes

Let's try it:

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix},$$

\*We are using here some standard notations from informal "set" language. If, speaking informally, we think of a "set" as something like a "bunch" or a "collection," then we write

$$\alpha \in S$$

to mean  $S$  is (more-or-less) a collection, and  $\alpha$  is one of the individual things included among the collection. If this mildly abstract language seems confusing, you can forget it.

The illustrative examples probably make clear what we mean by "algebraic closure." (See, if you wish, Appendix C, for a further discussion of "sets.")

and since this "answer" matrix is still in the form

$$\begin{pmatrix} \square & 0 \\ 0 & 0 \end{pmatrix},$$

this set of matrices is closed under addition.

We could ask, is the set of matrices

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

closed under multiplication?

Let's try it:

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Any two matrices  
of this form

↑  
yield a result of  
this same form  
(UV:  $0 \rightarrow A$ ).

Hence, this set is closed under multiplication.

We can keep exploring special sets of matrices in this same way. For example, is this set of matrices

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

closed under multiplication?

Let's try it:

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \times \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} = \begin{pmatrix} AD & 0 \\ 0 & BC \end{pmatrix}$$

Any two matrices  
of this form

↑  
yield a result which is  
not of this form.

Thus this set of matrices is *not* closed under multiplication.

(16) Jerrold says that you can match up numbers and matrices like this:

$$0 \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$2 \leftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$3 \leftrightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

What matrix would Jerrold match up with the number 7? with the number -1?

$$(16) 7 \leftrightarrow \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$-1 \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(17) Debbie says there is something special about Jerrold's matching:

If you add two numbers,

$$2 + 3 = 5,$$

you get a result that corresponds to adding the "matched" matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

(17) Yes, it does. For example:

Numbers

$$2 \times 3 = 6$$

Matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

Does Jerrold's "matching" also work out like this for multiplication?

(18) Nancy says that Jerrold's matching is what mathematicians call an isomorphism. Do you agree?

(19) Debbie says you can match numbers like this to get an isomorphism with respect to addition:

$$\begin{aligned} 1 &\leftrightarrow 2 \\ 2 &\leftrightarrow 4 \\ 3 &\leftrightarrow 6 \\ 4 &\leftrightarrow 8 \end{aligned}$$

What do you think?

(20) Nancy says Debbie's matching is also an isomorphism with respect to multiplication. Do you think Nancy is right?

(21) Charles made this up and claims it is a strange kind of isomorphism:

+	×
101	2
202	4
303	8
404	16

Do you see how it works?

(18) Nancy is right.

(19) Debbie is correct. Here is an example of how Debbie's isomorphisms work: since we have

$$\begin{aligned} 1 &\leftrightarrow 2 \\ 2 &\leftrightarrow 4, \end{aligned}$$

if we add the left-hand numbers

$$1 + 2,$$

the result should correspond to the result of adding the right-hand numbers:

$$2 + 4.$$

That is, we *should* have

$$1 + 2 \leftrightarrow 2 + 4.$$

The question is: do we? This would mean that we should expect the correspondence

$$3 \leftrightarrow 6,$$

and if we look at Debbie's table, we find that she does have 3 corresponding with 6, so the correspondence is an isomorphism. (Of course, checking one instance is not really a proof, but it does give the idea of how isomorphisms work.)

Try some other examples. Can you use variables to show that this isomorphism always works?

(20) Nancy is wrong. Here is a counterexample:

$$1 \times 2 \leftrightarrow 2 \times 4$$

In other words, 2 should correspond to 8 in our matching. Does it? No!

$$2 \leftrightarrow 4$$

So Debbie's matching is not an isomorphism *with respect to multiplication*.

(21) Charles is correct. This is a strange kind of isomorphism. Here is an example of how it works:

$$101 + 202 \leftrightarrow 2 \times 4 = 8$$

Does  $303 \leftrightarrow 8$  in our matching? Yes!

Can you see how the numbers in the second column were obtained? Can you use variables to show that this isomorphism always works? (Actually, what Charles has here is called a "table of logarithms," and it is useful for changing multiplication problems into addition problems.)

## MATRICES AND TRANSFORMATIONS

In Chapter 36 we studied transformations; in Chapters 37 through 39 we studied matrices. In Chapter 40, we bring the two ideas together. Bringing together ideas from different parts of mathematics is almost always a source of greater power and deeper insight.



### CHAPTER 40

## Matrices and Transformations

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(1) Ted has a method for using matrices to map a plane into a plane. Suppose he is using the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

He would start with a point—say, (1, 2)—and write it as a column matrix

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and multiply like this:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

He would say: The image of (1, 2) is (2, 4).

Can you find the image of (3, 1) using

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}?$$

(2) Can you give a geometric description of Ted's mapping?

## ANSWERS AND COMMENTS

(1) Notice that in Ted's method, the "input" point, with its coordinates written as a column matrix goes here:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} \phantom{1} \\ \phantom{2} \end{pmatrix}$$

↑  
Input

The "answer" (or "output" matrix or "image") goes here:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} \phantom{1} \\ \phantom{2} \end{pmatrix} = \begin{pmatrix} \phantom{2} \\ \phantom{4} \end{pmatrix}$$

↑  
Image point ("output" or "answer")

Hence, to find the image of (3, 1)—that is, to find

$T: (3, 1) \rightarrow ?$  (read:  $T$  maps the point (3, 1) into what?)—

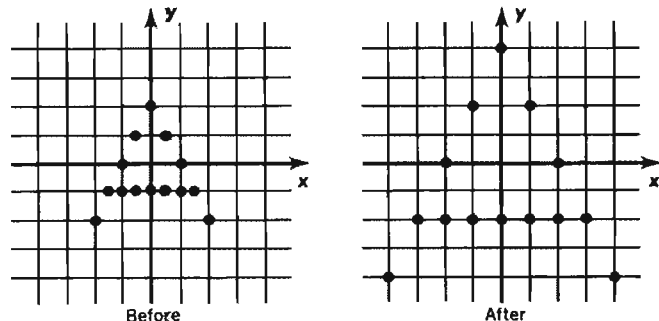
we would write

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

and so the image point is (6, 2):

$$T: (3, 1) \rightarrow (6, 2)$$

(2) Ted's mapping works like a "magnifying glass" or a "uniform stretching," by a factor of two. The point (0, 0) remains fixed, and every figure is doubled in size (as judged by linear dimensions). For example:



- (3) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

instead of

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

can you describe the mapping geometrically?

- (4) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

can you describe the mapping geometrically?

In the first and second quadrants, this is "Debbie's mapping." One could, therefore, use the present matrix mapping as an "extension" of Debbie's mapping, in the sense of Chapter 24.

- (5) If you use Ted's idea, but use the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

can you describe the mapping that you get?

- (3) Again, this is a "uniform magnification" or a "uniform stretching," but this time by a factor of 3. If you are in doubt, pick a few points outlining some simple figure, map each point as in question 1, and see how the image compares with the original figure.

- (4) Let's try out a few points, and see what seems to be happening:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so  $M: (0, 0) \rightarrow (0, 0)$ ; the point  $(0, 0)$  is not moved at all.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so  $M: (1, 0) \rightarrow (1, 0)$ ; the point  $(1, 0)$  is not moved at all.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix},$$

so  $M: (a, 0) \rightarrow (a, 0)$ ; any point on the  $x$ -axis is left unmoved.

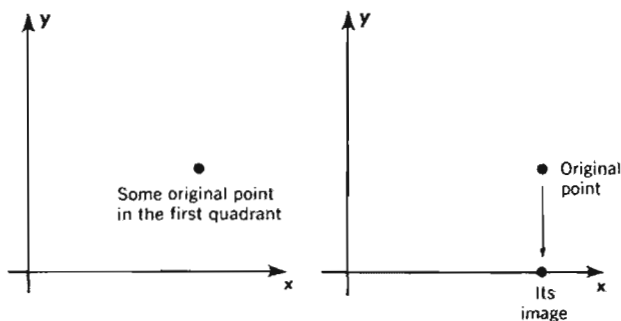
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so  $M: (0, 1) \rightarrow (0, 0)$ ; the point  $(0, 1)$  is mapped into  $(0, 0)$ .

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix},$$

so  $M: (a, b) \rightarrow (a, 0)$ ; any point is "projected" onto the  $x$ -axis.

This mapping is what mathematicians call "a projection onto the  $x$ -axis." For example, any point in the first quadrant will be mapped into the point on the  $x$ -axis directly beneath it.



What happens to points in the other quadrants? (See question 36, Chapter 36.)

- (5) Let's say the
- original point*
- (the "input") has the coordinates
- $(x_{\text{old}}, y_{\text{old}})$
- , and the
- image point*
- has the coordinates
- $(x_{\text{new}}, y_{\text{new}})$
- .

Then we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} x_{\text{old}} \\ y_{\text{old}} \end{pmatrix} = \begin{pmatrix} x_{\text{new}} \\ y_{\text{new}} \end{pmatrix},$$



and, by multiplying out these matrices, we get

$$x_{\text{new}} = y_{\text{old}},$$

$$y_{\text{new}} = x_{\text{old}}.$$

But this is precisely "Nancy's mapping," from question 45 of Chapter 36.

(6) What mapping do you get from this matrix?

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(6) As in our answer to question 5, we write

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} x_{\text{old}} \\ y_{\text{old}} \end{pmatrix} = \begin{pmatrix} x_{\text{new}} \\ y_{\text{new}} \end{pmatrix},$$

and we get

$$x_{\text{new}} = -x_{\text{old}},$$

$$y_{\text{new}} = -y_{\text{old}},$$

so that this is precisely "Ted's mapping" of question 47, Chapter 36.

(7) Make up some 2-by-2 matrices yourself, and see if you can find what kinds of geometric mappings your matrices produce?

(7) This can be a lot of fun. Have the students make figures (such as the letter "A," or Christmas trees, circles, squares, or whatever) and compare the figures "before" and "after." When in doubt, use the numerical coordinates and carry out the matrix multiplications. What happens is a bit like the "weird mirrors" you sometimes find in amusement parks.



## MATRICES AND SPACE CAPSULES

If the preceding chapters have seemed somewhat confusing, this chapter—which is fun and easy—may help to make sense out of these various ideas.



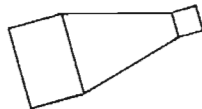
### CHAPTER 41

## Matrices and Space Capsules

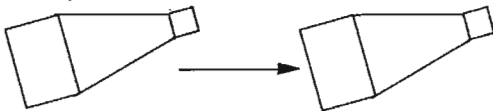
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You may find it easier to think about matrices and transformations if you know something about where they are used.

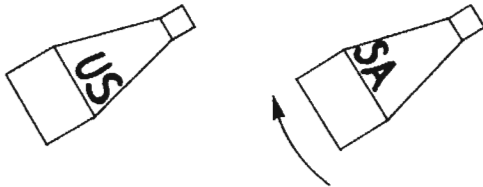
One example, very much in the spirit of the preceding chapter, comes from space science. Suppose we have a rocket or a space capsule (or, for that matter, an airplane)



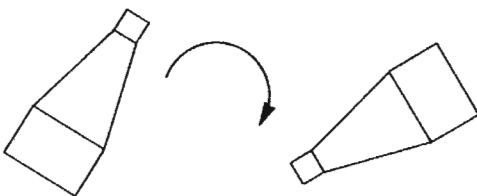
which is moving in space. Its motion can be very complicated. It can "move along a path from one spot to another,"



but—really at the same time as the motion above—it can also change its orientation (or, as it is known in space science, its attitude). For example, it can rotate like this:



Or it can "flop over" like this:



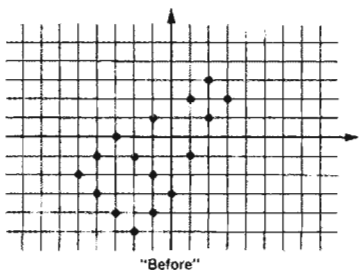
## ANSWERS AND COMMENTS

Now, it is essential to predict, and to observe, the motion of space capsules very precisely, using appropriate mathematics and high-speed digital computers.

The "flopping" kinds of motions are observed using "before" and "after" pictures of the kind we have just been studying. The "flopping" itself is regarded as a transformation, and is studied by means of its corresponding matrix.

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(1) Suppose a space capsule is represented by this set of points:



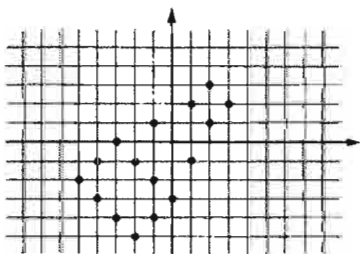
The set of points plotted on the graph above is the set  $\{(2, 1), (3, 2), (2, 3), (1, 2), (-1, 1), (-3, 0), (-4, -1), (-5, -2), (-4, -3), (-3, -4), (-2, -5), (-1, -4), (0, -3), (1, -1), (-2, -1), (-1, 2)\}$ .

Now, at this instant, a computer down on earth sends up a signal which causes the capsule to fire some small "flipping" rockets and "flop over." The computer on earth made a transformation using the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Assuming that the rockets all worked correctly, and the capsule did what the computer ordered, what is the new "position" or "attitude" of the space capsule?

(2) Suppose that the capsule started in this position



and the computer used the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$(1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} = \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix},$$

so

$$\begin{aligned} x_{new} &= y_{old}, \\ y_{new} &= -x_{old}. \end{aligned}$$

(Alternatively, you could avoid the use of variables here, and map one point at a time, using numbers:

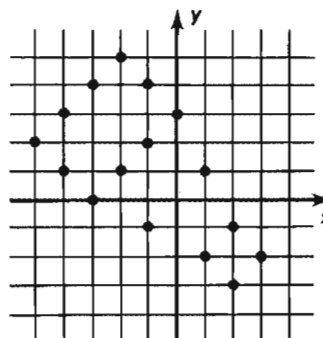
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

and so on.)

This gives us this set of points:

$$\{(1, -2), (2, -3), (3, -2), (2, -1), (1, 1), (0, 3), (-1, 4), (-2, 5), (-3, 4), (-4, 3), (-5, 2), (-4, 1), (-3, 0), (-1, -1), (-1, 2), (-2, 1)\}.$$

If we now plot these points, we get:

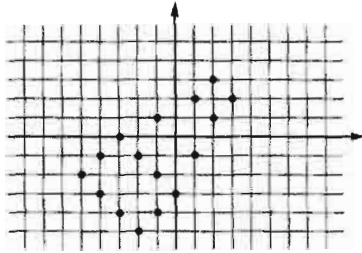


(2) and (3) We suggest you represent the capsule by actual number coordinates, as we did in question 1, and then use the same general method that we used there.

[page 149]

What would the capsule's position be after the maneuver was completed?

(3) Suppose the capsule started in this position

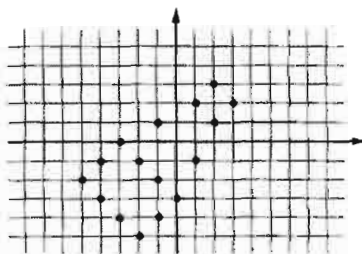


and the computer called for a "flipping" movement according to the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

How would the capsule look after this maneuver was completed?

(4) Suppose the capsule started in this position



and the computer called for a shift in attitude based on the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

But ... as soon as this maneuver was completed, someone discovered that the computer had made a mistake! Instead of calling for the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the computer ought to have called for a shift based on the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What matrix will get us back to where we ought to be?

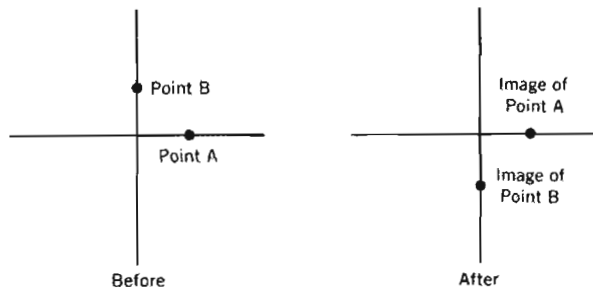
(4) The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

maps (1, 0) and (0, 1) as

$$\begin{aligned} M: (1, 0) &\longrightarrow (0, -1), \\ M: (0, 1) &\longrightarrow (1, 0) \end{aligned}$$

and represents a rotation through 90° clockwise.



So what has actually occurred has been a rotation through 90° clockwise.

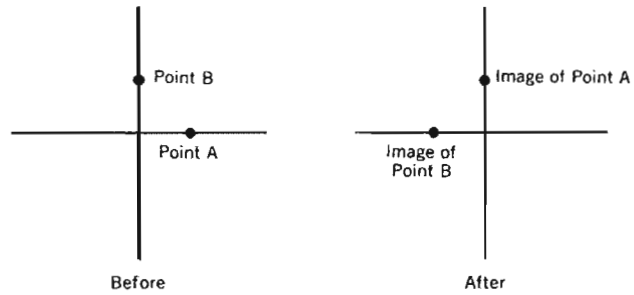
Now the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

maps (1, 0) and (0, 1) as

$$\begin{aligned} N: (1, 0) &\longrightarrow (0, 1), \\ N: (0, 1) &\longrightarrow (-1, 0) \end{aligned}$$

and represents a rotation through 90° counterclockwise.



So what *should have happened* was a rotation through 90° counterclockwise. To *get things back the way they ought to be*, then, we must rotate through 180° counterclockwise. This means we want to use the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

twice; but that is equivalent to using the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so we *shall use the matrix*

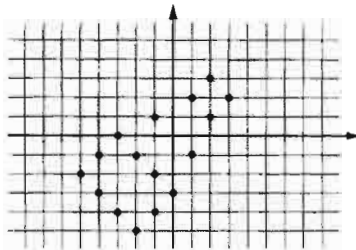
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

once.

You can verify this, if you wish, by working everything out with numbers, as in questions 1 through 3.

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(5) Suppose the space capsule started like this



and the computer called for a shift based on the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

After this maneuver was completed, the computer called for a shift based on the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(5) **After the fourth maneuver the capsule has rotated 90° counterclockwise.**

To understand this problem, let's pause and look at a little algebra. The first maneuver went like this:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} = \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} \quad (1)$$

We can write equation (1) in a more succinct notation:

the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ we can write as } M;$$

the column matrix

$$\begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \text{ we can write using the symbol } \vec{x}_{old}^* ;$$

\*Column matrices are often written using a letter with an arrow over it, as we have done here; they are frequently called column vectors.

After this second maneuver was completed, the computer called for a third, using the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

After the third maneuver was completed, the computer called for a fourth, using the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

What was the space capsule's position after the fourth maneuver had been completed?

the column matrix

$$\begin{pmatrix} x_{\text{new}} \\ y_{\text{new}} \end{pmatrix} \text{ we can write as } \vec{x}_n.$$

Then, the equation (1) becomes

$$M \cdot \vec{x}_0 = \vec{x}_n$$

Now, for the second maneuver let us write the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

as  $R$ , and the *new* coordinates as  $x'_n$ . Hence, we have

$$R \cdot \vec{x}_n = \vec{x}'_n.$$

For the third maneuver, we again use the matrix  $M$ ; let us call the new coordinates  $\vec{x}''_n$ . Hence, we can represent the third maneuver as

$$M \cdot \vec{x}'_n = \vec{x}''_n.$$

Finally, if we write the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as  $S$ , and the resulting new coordinates as  $\vec{x}'''_n$ , then we can write the *fourth maneuver* as

$$S \cdot \vec{x}''_n = \vec{x}'''_n.$$

Now, let's put all this together. Remember that our original coordinates are  $\vec{x}_0$  and our *final* coordinates are  $\vec{x}'''_n$ . Then we have

- |        |   |  |
|--------|---|--|
| (i)    | $S \cdot \vec{x}''_n = \vec{x}'''_n$                                  | Fourth maneuver                        |
| (ii)   | $M \cdot \vec{x}'_n = \vec{x}''_n$                                    | Third maneuver                         |
| (iii)  | $S \cdot (M \cdot \vec{x}_n) = \vec{x}'''_n$                          | PN, from line (i), using line (ii)     |
| (iv)   | $(S \cdot M) \cdot \vec{x}_n = \vec{x}'''_n$                          | ALM (which works also for matrices!)   |
| (v)    | $R \cdot \vec{x}_n = \vec{x}'_n$                                      | Second maneuver                        |
| (vi)   | $(S \cdot M) \cdot (R \cdot \vec{x}_n)$<br>$= \vec{x}'''_n$           | PN, from line (iv), using line (v)     |
| (vii)  | $[(S \cdot M) \cdot R] \cdot \vec{x}_n$<br>$= \vec{x}'''_n$           | ALM                                    |
| (viii) | $M \cdot \vec{x}_0 = \vec{x}_n$                                       | First maneuver                         |
| (ix)   | $[(S \cdot M) \cdot R] \cdot$<br>$(M \cdot \vec{x}_0) = \vec{x}'''_n$ | PN, from line (vii), using line (viii) |

$$(x) \quad \left\{ \left[ (S \cdot M) \cdot R \right] \cdot M \right\} \\ \cdot \vec{x}_0 = \vec{x}_n'''' \quad \text{ALM}$$

Hence, we see that the matrix

$$\left[ (S \cdot M) \cdot R \right] \cdot M$$

will take us from the first initial coordinates directly to the final coordinates. Let's see what the matrix is:

$$S \cdot M: \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(S \cdot M) \cdot R: \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[ (S \cdot M) \cdot R \right] \cdot M: \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hence, the single matrix which we shall call  $F$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

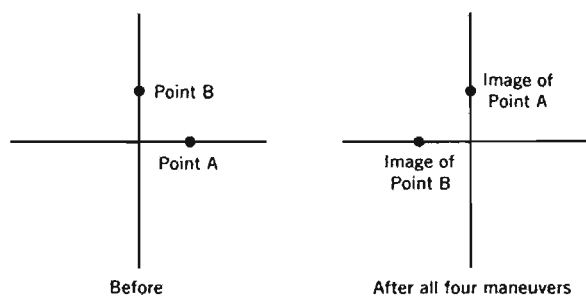
represents the result of all four maneuvers together! Now, what does

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

do?

Let's try (1, 0) and (0, 1):

$$F: (1, 0) \longrightarrow (0, 1) \\ F: (0, 1) \longrightarrow (-1, 0)$$



Consequently, the combined effect of all four maneuvers is a rotation  $90^\circ$  counterclockwise.

We can also do this geometrically, using almost no algebra! Here is how:

(a) The first maneuver used the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We saw (in question 1) that this corresponds to a rotation of  $90^\circ$  clockwise.

(b) The second maneuver used the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which [as you can easily verify for yourself by studying what happens to the points  $(1, 0)$  and  $(0, 1)$ ] corresponds to a rotation through  $180^\circ$  (for  $180^\circ$ , it doesn't matter to the final position whether we turn clockwise or counterclockwise).

(c) Putting the first two maneuvers together, we get  $90^\circ$  clockwise



then  $180^\circ$  counterclockwise



which amounts to  $90^\circ$  counterclockwise



(d) The third maneuver again used the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that is, another rotation  $90^\circ$  clockwise; hence the result of all three maneuvers thus far is to return to the original orientation



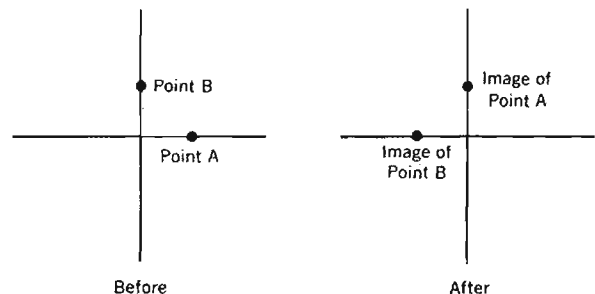
(e) Finally the fourth maneuver used the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which works like this:

$$S: (1, 0) \rightarrow (0, 1)$$

$$S: (0, 1) \rightarrow (-1, 0)$$



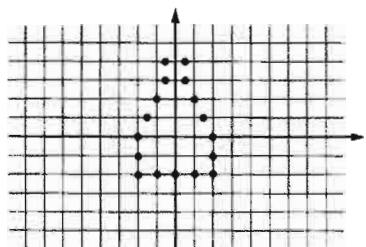
Thus this fourth maneuver is a rotation through  $90^\circ$  counterclockwise



This, then, is the final result of all four maneuvers together: a rotation counterclockwise through  $90^\circ$ .



(6) A space capsule went on a long flight, lasting seven months. At the beginning of the flight, the capsule's attitude was like this:



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During the flight, the computer called for 260 shifts, according to this list:

31 shifts, each based on the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

26 shifts, based on the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

203 shifts, based on the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What was the attitude of the space capsule after all of these maneuvers had been completed?

#### A TABLE OF MATRIX INVERSES

You may find it convenient to have this table of matrix inverses available in case you ever need to use it.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- (6) Putting all these shifts together, we have 31 rotations counterclockwise, each through an angle of  $90^\circ$ . Now, every four such rotations gets you back where we started; hence,  $31 - 28 = 3$ , and the result *thus far* is three rotations counterclockwise through  $90^\circ$ , which is equivalent to one rotation clockwise through  $90^\circ$ :



We then have 26 rotations clockwise, each through an angle of  $90^\circ$ . Again, four get you back where you started;  $26 - 24 = 2$ , so this is equivalent to one rotation through  $180^\circ$ . Combining with the preceding result gives us



the equivalent of one rotation through  $90^\circ$  counterclockwise:



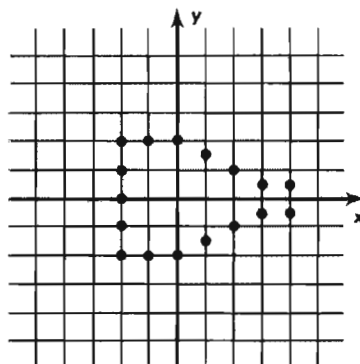
Then we consider the 203 rotations through  $180^\circ$ . Now, any two of these get you back where you started; hence,  $203 - 202 = 1$ ; this is equivalent to one rotation through  $180^\circ$ . Combining with our previous result, we have



which is equivalent to one rotation clockwise through  $90^\circ$ :



Hence, the *final* orientation is:



[page 152]

This table of matrix inverses will be helpful in Chapters 42 and 43.

$$\begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 12 \\ 2 & 2 \end{pmatrix} \times \begin{pmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \times \begin{pmatrix} 10 & 12 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 10 \\ 5 & 8 \end{pmatrix} \times \begin{pmatrix} -4 & 5 \\ \frac{1}{2} & -3 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ \frac{1}{2} & -3 \end{pmatrix} \times \begin{pmatrix} 6 & 10 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} = \begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} \times \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix} \times \begin{pmatrix} -1 & \frac{3}{5} \\ 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{5} \\ 1 & -\frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 & 8 \\ -2 & 3 & -10 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 8 \\ -2 & 3 & -10 \\ 1 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## SIMULTANEOUS EQUATIONS

In this chapter we cover a lot of ground. Although we deal with only one topic – solving simultaneous linear equations – our methodology develops rapidly. We begin by merely guessing. This method is valuable, and has been unjustly disparaged in the past. Unless we inhibit our students, they enjoy guessing; and “looking at the problem until you can find a clue that leads you to the answer” gives a very deep insight into the nature of the problem. If we let our students develop an expectation that they do not need to explore, that they can merely wait until we tell them the answers (or at least until we show them the method), then the students almost visibly wither before our eyes – after awhile, they never seem to discover or explore on their own; they just sit there and wait until we tell them.

The preceding paragraph attempts to defend our first method of approach in this chapter: by guessing (or, more accurately, by looking for clues). We quickly move on to other methods. The problems are arranged so that the students learn to turn a “simultaneous linear equation” problem into a “matrix” problem. They can solve the resulting “matrix” problem by using the Table of Matrix Inverses given on pages 151-152 of the student book. This approach should be easy enough if the students have had sufficient experience with matrices in the preceding chapters.

Finally, in question 15, we suggest a parallel between *simultaneous linear equations* and one equation in one unknown. If question 15 seems too sophisticated for the children in your classes, then please leave it out.



### CHAPTER 42

## Simultaneous Equations

[page 153]

- (1) These are known as “simultaneous equations”:

$$\begin{cases} \square + \triangle = 10 \\ \square - \triangle = 8 \end{cases}$$

The same number must go in both  $\square$ 's, and the same number in both  $\triangle$ 's. Can you find the  $\square$  number and the  $\triangle$  number to make both statements true?

- (2) Can you find the  $\square$  number and the  $\triangle$  number to make both statements true?

$$\begin{cases} \square + \triangle = 25 \\ \square - \triangle = 23 \end{cases}$$

## ANSWERS AND COMMENTS

In questions 1 through 5, the “method” is merely that of “looking carefully at the problem.”

$$\begin{aligned} (1) \quad 9 &\rightarrow \square \\ 1 &\rightarrow \triangle \end{aligned}$$

$$\begin{aligned} (2) \quad 24 &\rightarrow \square \\ 1 &\rightarrow \triangle \end{aligned}$$

(3) Can you make a numerical replacement for the variable  $A$  and a numerical replacement for the variable  $B$ , so that both statements will be true?

$$\begin{cases} A + B = 16 \\ A - B = 12 \end{cases}$$

(4) Can you make a numerical replacement for the variable  $x$  and a numerical replacement for the variable  $y$ , so that both statements will be true?

$$\begin{cases} x + y = 101 \\ x - y = 99 \end{cases}$$

(5) Can you find the truth set for this pair of simultaneous equations?

$$\begin{cases} A + (2 \times B) = 104 \\ A - (2 \times B) = 96 \end{cases}$$

(6) Can you find the truth set?

$$\begin{cases} (2 \times A) + (3 \times B) = 103 \\ (5 \times A) + (5 \times B) = 255 \end{cases}$$

$$\begin{aligned} (3) \quad & 14 \rightarrow A \\ & 2 \rightarrow B \end{aligned}$$

$$\begin{aligned} (4) \quad & 100 \rightarrow x \\ & 1 \rightarrow y \end{aligned}$$

$$\begin{aligned} (5) \quad & 100 \rightarrow A \\ & 2 \rightarrow B \end{aligned}$$

$$\begin{aligned} (6) \quad & 50 \rightarrow A \\ & 1 \rightarrow B \end{aligned}$$

This is the first problem where the students may be unable to guess the answer.

Here we start a more systematic method. First, we take our two equations:

$$\begin{aligned} (2 \times A) + (3 \times B) &= 103, \\ (5 \times A) + (5 \times B) &= 255. \end{aligned}$$

Then we rewrite them *in matrix notation*, as

$$\begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 103 \\ 255 \end{pmatrix}.$$

Next we look in the Table of Matrix Inverses to find the *multiplicative inverse* of

$$\begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix}.$$

which is

$$\begin{pmatrix} -1 & \frac{3}{2} \\ 1 & \frac{2}{3} \end{pmatrix}.$$

Now we "left-multiply" both sides of equation (1) by the inverse

$$\begin{pmatrix} -1 & \frac{3}{2} \\ 1 & \frac{2}{3} \end{pmatrix} \times \left[ \begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} \right] = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & \frac{2}{3} \end{pmatrix} \times \begin{pmatrix} 103 \\ 255 \end{pmatrix}.$$

Next, we use ALM:

$$\left[ \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & \frac{2}{3} \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix} \right] \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & \frac{2}{3} \end{pmatrix} \times \begin{pmatrix} 103 \\ 255 \end{pmatrix}.$$

Finally, we carry out the matrix multiplications

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -103 + \frac{1}{3} \times 255 \\ 103 - \frac{2}{3} \times 255 \end{pmatrix},$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 50 \\ 1 \end{pmatrix}$$

so that  $A = 50$  and  $B = 1$ .

(7) Debbie has a secret method for solving simultaneous equations. She used her secret method on the pair of simultaneous equations

$$\begin{cases} (2 \times A) + (3 \times B) = 14 \\ (4 \times A) + (5 \times B) = 26 \end{cases}$$

and she says the correct replacements are

$$\begin{aligned} 4 &\rightarrow A \\ 2 &\rightarrow B. \end{aligned}$$

Is Debbie right?

[page 154]

(8) Debbie explained her secret method like this: First, she took the equations

$$\begin{cases} (2 \times A) + (3 \times B) = 14 \\ (4 \times A) + (5 \times B) = 26 \end{cases}$$

and rewrote them as a problem in matrix multiplication:

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then, she looked in a table to find the multiplicative inverse of the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}.$$

What she found was

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}.$$

Then, she took this inverse, and wrote

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then she used the associative law for multiplication (ALM), and got

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

Then, she said

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so she wrote

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

(7) and (8) This is the same method that we used in answering question 6.

Finally, she carried out both of these matrix multiplications and got

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} (-\frac{3}{2} \times 14) + (\frac{3}{2} \times 26) \\ (2 \times 14) + (-1 \times 26) \end{pmatrix},$$

which is the same as

$$A = (-\frac{3}{2} \times 14) + (\frac{3}{2} \times 26) = -35 + 39 = 4$$

$$B = (2 \times 14) + (-1 \times 26) = 28 - 26 = 2.$$

Can you understand Debbie's "secret method"? Do you think you can use it to solve simultaneous equations?

(9) Try Debbie's "secret method" on this pair of simultaneous equations:

$$\begin{cases} (2 \times A) + (3 \times B) = 32 \\ (4 \times A) + (5 \times B) = 60 \end{cases}$$

Were you able to make it work?

$$(9) \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 32 \\ 60 \end{pmatrix}$$

The table tells us that the inverse matrix is

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix},$$

so we write:

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 32 \\ 60 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 32 \\ 60 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \times 32 + \frac{3}{2} \times 60 \\ 2 \times 32 - 1 \times 60 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$\begin{aligned} A &= 10 \\ B &= 4 \end{aligned}$$

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Can you solve these pairs of simultaneous equations?

$$(10) \begin{cases} (5 \times A) + (6 \times B) = 183 \\ (3 \times A) + (4 \times B) = 115 \end{cases}$$

$$(10) \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 183 \\ 115 \end{pmatrix}$$

The table tells us the inverse matrix is

$$\begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix},$$

so we write:

$$\begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 183 \\ 115 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \times 183 - 3 \times 115 \\ -\frac{3}{2} \times 183 + \frac{5}{2} \times 115 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 366 - 345 \\ -3 \times 91\frac{1}{2} + 5 \times 57\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$$

$$\begin{aligned} A &= 21 \\ B &= 13 \end{aligned}$$

$$(11) \begin{cases} (7 \times A) + (10 \times B) = 440 \\ (5 \times A) + (7 \times B) = 310 \end{cases}$$

$$(11) \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 440 \\ 310 \end{pmatrix}$$

The table tells us that the inverse matrix is

$$\begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix}$$

so we write:

$$\begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} \times \left[ \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} \right] = \begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} \times \begin{pmatrix} 440 \\ 310 \end{pmatrix}$$

$$\left[ \begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} \times \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \right] \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -7 \times 440 + 10 \times 310 \\ 5 \times 440 - 7 \times 310 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3080 + 3100 \\ 2200 - 2170 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$A = 20$$

$$B = 30$$

You can solve problems 12 through 14 by the same method.

$$(12) \begin{cases} (3 \times A) + (7 \times B) = 12 \\ (5 \times A) + (12 \times B) = 20\frac{1}{2} \end{cases}$$

$$(12) \begin{cases} A = \frac{1}{2} \\ B = \frac{3}{2} \end{cases}$$

Can you solve these systems of simultaneous equations?

$$(13) \begin{cases} (1 \times A) + (2 \times B) + (4 \times C) = 33 \\ (4 \times A) + (5 \times B) + (6 \times C) = 76 \\ (1 \times A) + (1 \times B) + (1 \times C) = 15 \end{cases}$$

$$(13) \begin{cases} A = 1 \\ B = 12 \\ C = 2 \end{cases}$$

Notice that in this problem the matrix equation is

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 33 \\ 76 \\ 15 \end{pmatrix}$$

$$(14) \begin{cases} A + (2 \times B) + (4 \times C) = 35 \\ (4 \times A) + (5 \times B) + (6 \times C) = 46 \\ A + B + C = 7 \end{cases}$$

$$(14) \begin{cases} A = 1 \\ B = 2 \\ C = 10 \end{cases}$$

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(15) Mary Frances says that Debbie's method for simultaneous equations is really like a method for solving "one equation in one unknown." To convince her friends, Mary Frances wrote this:

(15) I myself think Mary Frances' remark is interesting and valuable. Admittedly, it is a bit complicated.

The point here is to suggest a close parallel between *simultaneous linear equations* and the theory of one equation in one unknown.

School children may regard this mainly as an interesting curiosity, and we would leave it at that for the time being. Actually, it is a very suggestive curiosity, pointing toward the great value of the modern theory of linear operators on function spaces or linear spaces. At present "linear spaces" constitute a very advanced (and somewhat esoteric) mathematical topic, but one which may become much more familiar in the years ahead.

	Debbie's "Secret" Method	Mary Frances' "Shorthand" Way of Writing Debbie's Method	An example of "One Equation in One Unknown"
The original problem:	$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	Let $M$ stand for the matrix $\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix};$ let $\vec{u}$ (for "unknown") stand for the column matrix* $\begin{pmatrix} A \\ B \end{pmatrix};$ $\vec{K}$ (for "known") stand for the column matrix $\begin{pmatrix} 27 \\ 19 \end{pmatrix};$ then, the matrix problem can be written as: $M \times \vec{u} = \vec{K}.$	$3 \times N = 12$
Identify the coefficients of the "unknowns":	$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 27 \\ 19 \end{pmatrix}$ ↑ here	$M \times \vec{u} = \vec{K}$ ↑ here	$3 \times N = 12$ ↑ here
Find the multiplicative inverse for this coefficient:	$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$M \times M = I$ , where $I$ stands for the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{3} \times 3 = 1$
"Left-multiply" the original equation by this inverse:	$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \left[ \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} \right] = \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$M \times (M \times \vec{u}) = M \times \vec{K}$	$\frac{1}{3} \times (3 \times N) = \frac{1}{3} \times 12$
Use ALM:	$\left[ \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \right] \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$(M \times M) \times \vec{u} = M \times \vec{K}$	$(\frac{1}{3} \times 3) \times N = \frac{1}{3} \times 12$
Use the "inverse" property:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$1 \times \vec{u} = M \times \vec{K}$	$1 \times N = \frac{1}{3} \times 12$
Use the Law for 1, in appropriate form:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$	$1 \times \vec{u} = \vec{u}$	$1 \times N = N$
And get:	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 27 \\ 19 \end{pmatrix}$	$\vec{u} = M \times \vec{K}$	$N = \frac{1}{3} \times 12$
Complete any unfinished multiplications:	$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} (-7 \times 27) + (10 \times 19) \\ (5 \times 27) + (-7 \times 19) \end{pmatrix}$ $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -189 + 190 \\ 135 + 133 \end{pmatrix}$ $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{matrix} 1 \rightarrow A \\ 2 \rightarrow B \end{matrix}$ (or the truth set is $\{(1, 2)\}$ )	$\vec{u} = M \times \vec{K}$ is as far as you can go in the "shorthand" notation.	$N = 4$ $4 \rightarrow N$ (or, the truth set is $\{4\}$ )

Do you think Mary Frances has a good idea here, or not? What is wrong with her idea? Is there anything good about it?

\*We are using arrows over letters (as with  $\vec{u}$ ) to indicate column matrices.



## TAXIS, WIDGETS, AND ALPHA-BETA-GAMMA MIX

The purpose of this chapter is to give a few word problems that lead to systems of simultaneous equations; the problems, and especially the first problem, give some idea of the somewhat oblique way that mathematics relates to reality.

For example, in problem 1 there are no "numbers," or any other mathematics, in reality itself. In order to use mathematics we must talk about not reality itself, but simplified abstract "models" of reality.

To put this same idea another way; in problem 1 we are asked to determine a purchase of vehicles, sedans and station wagons, that will maximize a taxi company's profit in the year following the purchase.

Can we possibly "figure this out"? Obviously, we cannot — not in *actual reality*. To do so would require us (or the taxi company) to predict the future for one year in advance, and not even mathematics can predict the future. For example, it could turn out that, in the production of new cars, certain defects (this year!) might appear in station wagons but not in sedans. The taxi company could not know this in advance. Again, after the company bought station wagons, the town council might pass a law making it illegal to use station wagons as taxis and limousines. Again, the company might not be able to predict this in advance.

There are many other aspects of the future that the company cannot necessarily predict in advance: Will business for large groups, traveling in limousines, increase or decrease next year? Will any unusual services the company provides begin to "catch on" and create new demands for service? Will their accident record next year be better or worse than this year? (That can make a difference, since station wagons cost more than sedans.)

What, then, are we to do? We must make some *simplifying assumptions* if we are to make use of the data given us. Our conclusion will be valid *to the extent that* these assumptions are good approximations to reality, and only to this extent.

Additional problems (and easier ones) leading to systems of simultaneous equations can be found in *Discovery*, Chapter 46. Also, some interesting problems (arising in geometry) are given on pages 248, 249, 250, and 252 of Brumfiel (75). See also Beberman (87).



### CHAPTER 43

## Taxis, Widgets, and Alpha-Beta-Gamma Mix

[page 158]

(1) The Hurry-Up Taxi Company owns some sedans and some station wagons. In 1965 they owned 3 station wagons and 7 sedans. In 1966 they owned 5 station wagons and 12 sedans. In 1965 the company earned a

## ANSWERS AND COMMENTS

- (1) Let us assume, first, that the situation is not changing rapidly over a three-year period, so that we can assume things are the same in 1965, in 1966, and in 1967. Let us also assume that, averaged over an entire year, every sedan earns as much as

profit of \$53,560, after all expenses were paid. In 1966 the company earned a profit of \$91,000, after all expenses were paid.

For 1967 they can buy a few additional cars. Should they buy sedans or station wagons?

*every other sedan, and every station wagon earns as much as every other station wagon. With these assumptions, we can make use of the data given to us.*

*Suppose that the average earnings, after all expenses are paid, for one year for a sedan are represented by  $D$  dollars and that the annual earnings after expenses of a station wagon are  $W$  dollars.*

Then, for 1965, we have

$$(3 \times W) + (7 \times D) = 53,560,$$

and for 1966 we have

$$(5 \times W) + (12 \times D) = 91,000.$$

In other words, we have this system of simultaneous equations:

$$\begin{cases} (3 \times W) + (7 \times D) = 53,560 \\ (5 \times W) + (12 \times D) = 91,000. \end{cases}$$

In matrix form, this would be

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} W \\ D \end{pmatrix} = \begin{pmatrix} 53,560 \\ 91,000 \end{pmatrix}.$$

Using Debbie's method, we solve this as follows:

$$\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \left[ \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} W \\ D \end{pmatrix} \right] = \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 53,560 \\ 91,000 \end{pmatrix}$$

$$\left[ \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \right] \times \begin{pmatrix} W \\ D \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 53,560 \\ 91,000 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} W \\ D \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 53,560 \\ 91,000 \end{pmatrix}$$

$$\begin{pmatrix} W \\ D \end{pmatrix} = \begin{pmatrix} (12 \times 53,560) + (-7 \times 91,000) \\ (-5 \times 53,560) + (3 \times 91,000) \end{pmatrix}$$

$$\begin{pmatrix} W \\ D \end{pmatrix} = \begin{pmatrix} 5720 \\ 5200 \end{pmatrix}$$

In other words, *after paying all expenses*, a sedan earns a profit of \$5200 per year, or (on the average) \$100 per week, whereas a station wagon earns \$110.

If the company can buy the same number of new station wagons as they could sedans (charging off the higher wagon cost as part of the "expenses already paid" before reporting these figures), then they will be better off to buy all new wagons, getting no new sedans. If they could buy *more* sedans, then they must balance this against the fact that wagons earn 10% more *after all expenses are paid*. (This might depend upon how the company went about the task of raising additional capital.)

(2) The Acme Widget Company sells widgets. However, they also sell wigglyups. On the day before Christ-

(2) One way to tackle this problem uses simultaneous equations and "Debbie's method."

mas they were hurrying to get all their orders packed into cartons, sealed, addressed, and shipped off.

Somebody found two cartons which had been packed and sealed, but not labeled! What was in them?

Well, they were either the order for Smith's Department Store or the order for Edward's Emporium. But which were they?

The Smith's order called for 2 cartons, one containing 7 widgets and 10 wigglyups and the other containing 5 widgets and 7 wigglyups.

The Edward's order called for 2 cartons, one containing 6 widgets and 10 wigglyups, the other containing 5 widgets and 8 wigglyups.

One man suggested weighing the cartons. They did and found that the first carton weighed 59.4 pounds. The second carton weighed 41.8 pounds.

Larry, one of the men in the shipping room, said, "Now I know; this must be the order for Smith's!"

Bill, another shipping room man, said, "I'm sorry, Larry, old fellow, but you're wrong! That must be the order for Edward's!"

Was either man right? What do you think?

(3) The Cochran Chemical Company had a large supply of three chemicals, which we'll call alphathane, betathane, and gammathane.

They also had a District Manager who wrote all of his records on old envelopes and then usually lost the envelopes.

The First Vice-President telephoned the District Manager and asked how much alphathane, betathane, and gammathane he had in his storage spaces.

But the District Manager had written the amounts on an envelope, and he couldn't find the envelope. Then the District Manager spotted an old envelope where he had worked out some calculations, and he read this to the First Vice-President:

Let  $D$  be the weight of one widget and  $G$  be the weight of one wigglyup. If this is the Smith order, we have

$$\begin{cases} (7 \times D) + (10 \times G) = 59.4 \\ (5 \times D) + (7 \times G) = 41.8 \end{cases}$$

which can be written

$$\begin{pmatrix} 7 & 10 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} D \\ G \end{pmatrix} = \begin{pmatrix} 59.4 \\ 41.8 \end{pmatrix}$$

Using Debbie's method, we get

$$\begin{pmatrix} D \\ G \end{pmatrix} = \begin{pmatrix} -7 & 10 \\ 5 & -7 \end{pmatrix} \times \begin{pmatrix} 59.4 \\ 41.8 \end{pmatrix}$$

$$D = (-7 \times 59.4) + (10 \times 41.8) = 2.2$$

$$G = (5 \times 59.4) + (-7 \times 41.8) = 4.4$$

This seems to be consistent with the data given to us. Apparently the order could be for Smith's!

But wait! This does not yet settle the matter! Perhaps the Edward's data will also prove to be possible! If the order is for Edward's, we have

$$\begin{cases} (6 \times D) + (10 \times G) = 59.4 \\ (5 \times D) + (8 \times G) = 41.8 \end{cases}$$

$$\begin{pmatrix} 6 & 10 \\ 5 & 8 \end{pmatrix} \times \begin{pmatrix} D \\ G \end{pmatrix} = \begin{pmatrix} 59.4 \\ 41.8 \end{pmatrix}$$

$$\begin{pmatrix} D \\ G \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ \frac{1}{2} & -3 \end{pmatrix} \times \begin{pmatrix} 59.4 \\ 41.8 \end{pmatrix}$$

$$D = (-4 \times 59.4) + (5 \times 41.8) = 209.0 + -237.6$$

Without going further, we see that

$$D < 0,$$

which is clearly impossible. This *cannot* be the order for Edward's!

- (3) Suppose the district manager has  $A$  tons of alphathane,  $B$  tons of betathane, and  $C$  tons of gammathane. If he used *all* of his alphathane,  $A$ , plus twice his amount of betathane,  $A + (2 \times B)$ , plus four times his amount of gammathane,  $A + (2 \times B) + (4 \times C)$ , he would end up with 370 tons of soft mix,

$$A + (2 \times B) + (4 \times C) = 370.$$

If he used four times as much alphathane as he has,  $(4 \times A)$ , plus five times as much betathane as he has,  $(4 \times A) + (5 \times B)$ , plus six times as much gammathane as he has,  $(4 \times A) + (5 \times B) + (6 \times C)$ , he would end up with 880 tons of ordinary mix,

$$(4 \times A) + (5 \times B) + (6 \times C) = 880.$$

"If I wanted to make 370 tons of soft alpha-beta-gamma mix, I would need all of the alphathane I have, [page 159]

plus twice as much betathane as I have, plus 4 times as much gammathane as I have. If I wanted to make 880 tons of ordinary alpha-beta-gamma mix, I would need 4 times as much alphathane as I have, 5 times as much betathane as I have, and 6 times as much gammathane as I have. But, on the other hand, I could mix together all the alphathane, betathane, and gammathane that I have, and I would get 180 tons of hard alpha-beta-gamma mix. Does that answer your question?"

Does it?

If he had put all of his alphathane,  $A$ , plus all of his betathane,  $A + B$ , plus all of his gammathane,  $A + B + C$ , together, he would have had 180 tons of hard mix;

$$A + B + C = 180.$$

Consequently, we have the system

$$\begin{cases} (1 \times A) + (2 \times B) + (4 \times C) = 370 \\ (4 \times A) + (5 \times B) + (6 \times C) = 880 \\ (1 \times A) + (1 \times B) + (1 \times C) = 180 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 370 \\ 880 \\ 180 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & -2 & 8 \\ -2 & 3 & -10 \\ 1 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} 370 \\ 880 \\ 180 \end{pmatrix}$$

$$A = (1 \times 370) + (-2 \times 880) + (8 \times 180) = 50$$

$$B = (-2 \times 370) + (3 \times 880) + (-10 \times 180) = 100$$

Since

$$A + B + C = 180,$$

we must have

$$C = 30.$$

Consequently, the district manager has—or, perhaps, the ex-district manager had—50 tons of alphathane, 100 tons of betathane, 30 tons of gammathane.

## NEW WAYS OF WRITING OLD NUMBERS

Before teaching this chapter, you may want to view the film entitled "Complex Numbers via Matrices."



### CHAPTER 44

## New Ways of Writing Old Numbers

[page 160]

(1) Dan made up a number system with numbers like  $\alpha, \beta, \gamma, \delta, \epsilon, \lambda, \nu, \mu, \dots$ . Dan's system worked like this:

$$\square \times \triangle = \triangle \times \square$$

$$\square + \triangle = \triangle + \square$$

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

$$\square \times (\triangle \times \nabla) = (\square \times \triangle) \times \nabla$$

$$\square + (\triangle + \nabla) = (\square + \triangle) + \nabla$$

$$\square \times \alpha = \alpha$$

$$\square + \alpha = \square$$

$$\square \times \beta = \square$$

$$\beta + \beta = \gamma$$

$$\gamma + \beta = \delta$$

$$\delta + \beta = \epsilon$$

Is Dan's system really a new system or is it just a new way of writing an old system?

## ANSWERS AND COMMENTS

(1) **Actually, Dan's description is not complete, but as far as it goes it appears to be a description of the nonnegative integers.**  
Dan's identity

$$\square + \alpha = \square$$

suggests that  $\alpha$  is really another name for our old friend zero. This interpretation is also consistent with Dan's identity

$$\square \times \alpha = \alpha,$$

which appears, then, to be saying

$$\square \times 0 = 0.$$

Dan's identity

$$\square \times \beta = \square$$

suggests that  $\beta$  is really just a new name for our old friend 1. In that case, Dan's statement

$$\beta + \beta = \gamma$$

indicates that  $\gamma$  is merely a new name for our old friend 2. Our verdict would be that this does **not** appear to be a new mathematical system. Dan has merely given **new names** to a familiar old system.

We could express this by saying that *there exists an isomorphism* between Dan's  $\{\alpha, \beta, \gamma, \sigma, \dots\}$  and the set of nonnegative integers  $\{0, 1, 2, 3, \dots\}$ . Now, as we have seen earlier, an *isomorphism* is a one-to-one correspondence, in this case between  $\{\alpha, \beta, \gamma, \sigma, \dots\}$  and  $\{0, 1, 2, 3, \dots\}$ :

$$\begin{array}{l} \alpha \leftrightarrow 0 \\ \beta \leftrightarrow 1 \\ \gamma \leftrightarrow 2 \\ \sigma \leftrightarrow 3 \\ \vdots \\ \vdots \end{array}$$

In this case, the correspondence has the property of "preserving" sums and products. For example:

$$\begin{array}{lll} \alpha \times \beta = \alpha & \alpha \leftrightarrow 0 & 0 \times 1 = 0 \\ & \beta \leftrightarrow 1 & \\ \\ \beta + \gamma = \delta & \beta \leftrightarrow 1 & \\ & \gamma \leftrightarrow 2 & 1 + 2 = 3 \\ & \delta \leftrightarrow 3 & \end{array}$$

We could also say this by saying that if  $S(\alpha, \beta, \gamma, \dots)$  is any statement involving  $\alpha, \beta, \gamma, \dots$ , and if  $S(0, 1, 2, \dots)$  is the corresponding statement involving  $0, 1, 2, \dots$ , then  $S(\alpha, \beta, \gamma, \dots)$  is true if and only if  $S(0, 1, 2, \dots)$  is true,

$$S(\alpha, \beta, \gamma, \dots) \Leftrightarrow S(0, 1, 2, \dots).$$

Some descriptions are "complete" in the sense that they specify exactly one thing. Other descriptions are "incomplete" in the sense that they *might* refer to one thing, or perhaps to a second, and so on.

Thus, "Mrs. Brown" is an *incomplete* description: you cannot be sure *who* is meant, since there are surely several people who fit this description.

"The tall lady with the red hair and blue eyes" is an *incomplete* description, since surely there are several of these.

"Miss Cynthia Parsons, who in September, 1964, lived in an apartment house at the corner of Massachusetts Avenue and Beacon Street in Boston, Massachusetts, U.S.A." is probably a complete description, since it probably identifies *exactly* one person.

Now, in mathematical logic, a description which describes *exactly one thing* (which will usually mean "exactly one mathematical system") is called *categorical*.

Obviously, as the examples above show, the ordinary work of the world is handled by using *incomplete* descriptions. *Complete* descriptions are usually long and awkward.

The same is true in mathematics, and in this book we shall use *incomplete* descriptions of mathematical systems -- that is, descriptions which are not *logically categorical*. For example, Dan's description is not categorical.

That means we have to agree to give one another credit for generally good intentions: since Dan's description is not categorical, you can't tell, for sure, just what mathematical system he is talking about. Hence, you can't be *sure* whether it's a "new" one, or just a renamed "old" one.

But just as we do when someone says "the lady in the brown coat," you have to assume that we mean more or less what we *appear* to mean. We are not "throwing any curves" -- at least, most of the time, we are not.

(2) Sarah says Dan's system is really an old system. Dan just writes it a new way. Sarah says she knows what  $\alpha$  really is. Do you?

(3) In Dan's system, what is  $\beta$ ? How do you know?

(4) In Dan's system, what is  $\gamma$ ?

(2)  $\alpha$  is merely Dan's way of writing zero.

(3)  $\beta$  is merely Dan's way of writing one.

(4)  $\gamma$  is merely Dan's way of writing two.

(5) Is Dan's system really a new system, or is it really a new way of writing an old system?

(6) Ellen made up a number system, with numbers like  $\phi$ ,  $\xi$ ,  $\mu$ ,  $\zeta$ ,  $\theta$ ,  $\kappa$ ,  $\tau$ ,  $\psi$ ,  $\dots$ . Ellen's system worked like this:

$$\square \times \triangle = \triangle \times \square$$

$$\square + \triangle = \triangle + \square$$

$$\square + (\triangle + \nabla) = (\square + \triangle) + \nabla$$

$$\square \times (\triangle \times \nabla) = (\square \times \triangle) \times \nabla$$

$$\square \times (\triangle + \nabla) = (\square \times \triangle) + (\square \times \nabla)$$

$$\square + \phi = \square$$

$$\square \times \xi = \xi$$

$$\square \times \phi = \square$$

$$\phi \neq \xi$$

$$\phi + \xi = \xi$$

$$\xi + \xi = \mu$$

$$\mu + \xi = \zeta$$

$$\zeta + \xi = \theta$$

$$\vdots$$

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Is Ellen's system really a new system or is it a new way of writing an old system?

(7) Jerry says that Ellen's system is really an old system. Jerry says that  $\phi$  is really 0,  $\xi$  is really 1,  $\mu$  is really 2, and so on. Do you agree?

(8) Martha says that Ellen's system is really a new system, because  $\phi$  is somewhat like 0, but not entirely like 0. What do you think?

(9) Louis made up a method of writing numbers using 2-by-2 matrices. How do you suppose Louis wrote 0?

(10) How do you suppose Louis wrote 1?

(11) How do you suppose Louis wrote 2?

(5) It appears to be our old friend, the system of nonnegative integers

$$\{0, 1, 2, \dots\}.$$

(6) At first glance, we see

$$\square + \phi = \square,$$

which can refer to a familiar old system only if  $\phi$  is a new name for zero. But we also have

$$\square \times \phi = \square,$$

which appears (at first glance) to require that  $\phi$  be a name for one.

These two identities, considered together, seem to say that Ellen's system *must be a new one*, quite unlike any we've seen before.

But, there is a catch here: Perhaps  $\phi$  is a name for zero, and *all Ellen's other symbols also name zero*. This will make everything come out all right, and we shall have described the system which contains only zero. But there is one obstacle to this interpretation, namely, Ellen's statement

$$\phi \neq \xi,$$

which says that  $\phi$  *cannot* name the same thing that  $\xi$  names.

Putting all of this together, we conclude that Ellen's system *must be a new one*.

(7) No. Jerry's interpretation will not work. It would require

$$\begin{array}{lll} \square \times \xi = \xi & \text{to mean} & \square \times 1 = 1, \\ \square \times \phi = \square & \text{to mean} & \square \times 0 = \square, \\ \phi \neq \xi & \text{to mean} & 0 \neq 1. \end{array}$$

These statements cannot be reconciled with any of our previous mathematical systems.

(8) Martha is right.

(9) Louis wrote zero as  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

(10) Louis wrote 1 as  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

(11) Louis wrote 2 as  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ .

(12) How would  $2 \times 3 = 6$  be written, in Louis's system?

$$(12) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

(13) How would  $2 + 3 = 5$  be written, in Louis's system?

$$(13) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

(14) How would Louis write  $-4$ ?

$$(14) \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

(15) How would Louis write the integer  $A$ ?

$$(15) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

Note: The correspondence

$$0 \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$2 \leftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ where } A \text{ is any integer}$$

is an *isomorphism*. This is the meaning of questions 12 and 13, and other similar questions.

(16) Bernice says that Louis has set up an **isomorphism** between our usual numbers and a subset of the set of 2-by-2 matrices. What do you think?

(16) **Bernice is correct.**

(17) What subset of the set of 2-by-2 matrices did Louis use?

$$(17) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

**If you assume  $A$  is an integer, this system of matrices is isomorphic to the system of integers. If you require  $A$  merely to be a rational number, this system of matrices is isomorphic to the system of rational numbers.**

(18) Under Louis's isomorphism, what "old-fashioned" number corresponds to the following matrix?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(18) **None. This matrix does not appear in Louis's system.**



## THE HESITANT SEARCH FOR NEW NUMBERS

Before teaching this lesson, you may want to view either the film "Solving Equations with Matrices" or the film "Complex Numbers via Matrices." Another relevant film, showing small-group instruction with a class of sixth-grade children is entitled "Small-Group Instruction: Committee Report on Rational Approximations."



### CHAPTER 45

## The Hesitant Search for New Numbers

[page 162]

The origin of the basic ideas of counting surely dates back to quite early prehistoric times. Of course, as we have seen, our present method of writing the numbers that we use in counting (that is to say, 1, 2, 3, . . .) comes from the Hindus. Our method may have been introduced to France and Italy, by way of Spain, by Pope Sylvester II, who, as a young man, had studied in Spanish schools run by the Moslems. The date for this is about 1000 A.D.

However, it appears that the method of writing

1, 2, 3, 4, . . .

was perhaps the only number idea that Pope Sylvester II brought back from Spain. He appears not to have brought back the important idea of zero, although the Hindus had conceived the idea of zero at least as early as 800 A.D.

There is a strange theme of searching and rejecting that threads through the history of mathematics, from the ancients down until nearly the present day: this is the hesitant search for new kinds of numbers.

If, in fact, you know only the "counting" numbers,

1, 2, 3, 4, . . .,

then what do you do when you want to cut a pie, or divide up a candy bar, or give number names to all of the points on the number line?

This, and similar problems, led men to invent numbers such as

$\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $2\frac{1}{3}$ ,  $\frac{2}{3}$ ,

and so on.

## ANSWERS AND COMMENTS

In 1484 the French mathematician Chuquet worked out many ideas about exponents. He came to recognize the role of negative integers as exponents, which we have seen in such problems as

$$\frac{x^3}{x^7} = x^{-4}.$$

Over a century earlier, Nicole Oresme (born in Normandy about 1323; died in 1382) had worked with the number line:



Both of these notions create a natural role for negative numbers, but their introduction and acceptance were gradual. People needed them, but they "didn't really believe in them."

For example, in 1544 the German mathematician Michael Stifel (1486-1567) published a volume entitled *Arithmetica Integra*. In this book, Stifel recorded the lines of "Pascal's triangle" as far as the line for

$$(R + S)^7 = R^7 + 7R^6S + \dots + S^7.$$

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He used letters to represent "unknowns," and used the modern symbols + (for addition), - (for subtraction), and  $\sqrt{\quad}$  (for square root). However, when Stifel encountered negative numbers as elements of the truth set for an open sentence, he rejected them, apparently not considering them "really appropriate," or something of the sort.\*

In 1572 Rafael Bombelli worked effectively with negative numbers, and appeared to have a considerable understanding of them.

But, even after you have the counting numbers<sup>†</sup>

1, 2, 3, 4, ...

and also zero

0, 1, 2, 3, 4, ...

and also fractions and "mixed numbers"

0, 1, 2,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $2\frac{1}{2}$ , ...

and also negative numbers.

0, 1, 2,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{2}{3}$ ,  $2\frac{1}{2}$ , -1, -2,  $-\frac{1}{2}$

you still encounter the need for new kinds of numbers.

Why?

\*Compare Eves, *op. cit.*, p. 219

†Nowadays, in the twentieth century, you can find different books by different authors that use the words "counting numbers" to have different meanings. This, of course, is inevitable. Not all authors agree, no matter what the topic under discussion may be. In particular, one will nowadays sometimes see the words "counting numbers" used to refer to the elements of the set {1, 2, 3, 4, ...} and in other books these words will refer to the elements of the set {0, 1, 2, 3, 4, ...}. If we construe that "counting" refers to the process most of us use when we "count on our fingers," then it seems reasonable to say that the counting numbers are 1, 2, 3, 4, ...; on the other hand, as is sometimes convenient, if we choose to regard the counting numbers as the ordinary answers to questions of the form "how many?" then it is reasonable to assume that we are talking about the set {0, 1, 2, 3, 4, ...}, since it may well happen that when someone says "How many brothers do you have?" the answer will turn out to be 0. Some people wish that all books were in complete agreement, but there is reason to feel that as long as life goes on this will not occur, and perhaps it is a good thing that it won't.

We have said earlier that the ancients, and the early Renaissance mathematicians, had worked out the general solution of the general quadratic equation. That was so, in a purely "procedural" sense that paralleled our own derivation of the solution of the general quadratic equation. They knew what to do—provided it worked out satisfactorily!

Now, there is one step in the procedure that will sometimes fail to work out satisfactorily: this is the process of taking the square root.

This can be written either as

$$x^2 - Ax + B = W$$

$$\left\{ \frac{A}{2} + \sqrt{W - B + \left(\frac{A}{2}\right)^2}, \frac{A}{2} - \sqrt{W - B + \left(\frac{A}{2}\right)^2} \right\}$$

or else as

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If

$$W - B + \left(\frac{A}{2}\right)^2$$

is a perfect square, such as 16 or 49 or 121, there is no difficulty. But there are two cases in which there [page 164] are difficulties—difficulties which early Renaissance mathematicians found quite serious. One case occurs if we encounter square roots such as

$$\sqrt{2} \quad \text{or} \quad \sqrt{45} \quad \text{or} \quad \sqrt{11},$$

and so on.

The other case occurs if we encounter square roots such as

$$\sqrt{-1} \quad \text{or} \quad \sqrt{-4} \quad \text{or} \quad \sqrt{-49},$$

and so on.

(1) What can you say about these two kinds of square roots (for example,  $\sqrt{2}$  versus  $\sqrt{-4}$ )?

(1) In the case of seeking numbers whose square is 2, we can "come close." For example:

$1^2 = 1$	Too small
$1.4^2 = 1.96$	Too small, but very much closer
$1.41^2 = 1.9881$	Too small, but quite close!
$1.414^2 = 1.999396$	Too small, but wouldn't you call that close?
$2^2 = 4$	Too big
$1.5^2 = 2.25$	Too big, but much closer
$1.42^2 = 2.0164$	Too large, but close!
$1.415^2 = 2.002225$	Too large, but surely close!

Continuing this process, we can get as close as you like (although one can prove that we shall never get *exactly* 2).

Now, the case of

$$\square \times \square = -4$$

is entirely different. Every number we know has a nonnegative square; hence, we cannot get any "closer" to  $-4$  than to use

$$0^2 = 0.$$

- (2) What can you say about the truth set for the open sentence

$$x^2 = -4?$$

- (3) Here is a research problem. Using the isomorphism

$$A \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

to guide you, develop some new numbers, so that you will be able to solve the equation

$$x^2 = -4.$$

If you check for algebraic closure under addition and under multiplication, you can work out an entirely new mathematical system. (Incidentally, the system you make up here is one which Descartes encountered, but he rejected it as not making sense. Today it is one of the most important mathematical systems that we know about.)

In this case, we cannot even get close! Nonetheless, it is actually easier to handle equations like

$$x^2 = -4$$

than it is to handle equations like

$$x^2 = 2.$$

- (2) Using numbers we already know (i.e., rational numbers), this truth set is empty.

- (3) The equation

$$x^2 = -4$$

becomes, when translated into matrix language, the equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}.$$

We'll leave this for you and your class to study. It's really a rather exciting problem.

## DETERMINANTS

Be careful not to confuse *determinants* with *matrices*. They are quite different. Note in particular that, whereas a matrix is really a whole array of different numbers, a determinant is merely a fancy way of writing a single number.



### CHAPTER 46 Determinants

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Don made up a mathematical system, like a game, by making up some rules.

(a) Don would begin by writing 4 numbers, like these:

$$\begin{array}{cc} 7 & 8 \\ 1 & 2 \end{array}$$

(b) Don **did not** want these to be the same thing as a **matrix**, because he already knew about matrices, and he wanted this to be a **new system**. So Don **did not** write:

$$\begin{pmatrix} 7 & 8 \\ 1 & 2 \end{pmatrix}$$

Instead he used **straight lines** and wrote:

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix}$$

(c) Don said, "Whenever I write

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix}$$

what I really mean is

$$(7 \times 2) - (8 \times 1)."$$

(1) What number does Don mean when he writes

$$\begin{vmatrix} 7 & 8 \\ 1 & 2 \end{vmatrix} ?$$

(2) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} ?$$

(3) What number does Don mean when he writes

$$\begin{vmatrix} 3 & 2 \\ 15 & 10 \end{vmatrix} ?$$

## ANSWERS AND COMMENTS

$$(1) (7 \times 2) - (8 \times 1) = 14 - 8 = 6$$

$$(2) (1 \times 4) - (2 \times 3) = 4 - 6 = -2$$

$$(3) (3 \times 10) - (2 \times 15) = 30 - 30 = 0$$

A very fancy method for writing zero.

- (4) What number does Don mean when he writes

$$\begin{vmatrix} 4 & 2 \\ 3 & 8 \end{vmatrix} ?$$

(5) Jane doesn't know Don's system, but she does know all about how variables work. Can you use variables to show Jane exactly how Don's system works?

- (6) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 1 \\ 3 & 7 \end{vmatrix} ?$$

- (7) What number does Don mean when he writes

$$\begin{vmatrix} 1 & 1 \\ 7 & 3 \end{vmatrix} ?$$

- (8) What number does Don mean when he writes

$$\begin{vmatrix} 3 & 7 \\ 1 & 1 \end{vmatrix} ?$$

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- (9) What number does Don mean when he writes

$$\begin{vmatrix} 1796 & 301 \\ 255 & 186 \end{vmatrix} ?$$

- (10) What number does Don mean when he writes

$$\begin{vmatrix} 301 & 1796 \\ 186 & 255 \end{vmatrix} ?$$

(11) Sandy's father says that somebody else, who called these things determinants, already invented Don's system before Don did.

(12) Alice says that Don's system doesn't look like it will ever be good for anything. What do you think?

$$(4) (4 \times 8) - (2 \times 3) = 32 - 6 = 26$$

- (5)

		<b>Paper 1,000,372-WJ</b>		
$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$	$=$	$(A \times D)$	$-$	$(B \times C)$

$$(6) (1 \times 7) - (1 \times 3) = 7 - 3 = 4$$

$$(7) (1 \times 3) - (1 \times 7) = 3 - 7 = -4$$

$$(8) (3 \times 1) - (7 \times 1) = 3 - 7 = -4$$

What happens when you reverse the columns of a 2-by-2 determinant? Suppose you reverse the rows?

$$(9) 1796 \times 186 - 301 \times 255 = 257,301$$

$$(10) \text{ Similarly, } 301 \times 255 - 1796 \times 186 = -257,301.$$

(11)-(12) **Sandy's father is correct. Wait until Chapter 47 to see whether determinants will be useful.**

## MATRIX INVERSES: A RESEARCH PROBLEM

Among the most important tools of contemporary applied mathematics are the various methods for computing matrix inverses. Some of the greatest mathematicians of the twentieth century have spent time trying to devise effective methods for solving this problem, particularly by using electronic computers.

What we do in this chapter is, of course, quite elementary – but at least we are working on the same problem that great mathematicians have worked on in recent years.



### CHAPTER 47

### Matrix Inverses:

### A Research Problem

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As we saw in Chapter 33, Professor George Polya of Stanford University has tried to describe some of the methods that scientists and mathematicians use in research.

We can try to practice some of these methods ourselves. First, we need a problem to work on. Here is one:

**Problem:** If you are given any 2-by-2 matrix, say

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

can you find a matrix

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}?$$

That is to say, if you are given a 2-by-2 matrix, can you find its multiplicative inverse?

**Suggestion:** Can you read about this problem somewhere?

**Answer:** Actually, you could read quite a bit about this problem. However, that may not be necessary just yet. Here is one problem that somebody else has already figured out:

If you are given the matrix

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix},$$

### ANSWERS AND COMMENTS

Presumably, if your students study carefully the illustrative example,

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

they will notice a peculiar pattern. Does this same pattern always work the same way? We'll try it and see. What we are seeking is a matrix

$$\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

such that

$$\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There are two easy ways to solve this problem. For the first method, the pattern of the illustrative example seems to work as follows. For the matrix

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix},$$

the inverse is found by taking the upper left-hand number and writing it in the lower right-hand spot:

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix}$$

its inverse is

$$\begin{pmatrix} 12 & 7 \\ -5 & 3 \end{pmatrix}$$

That is to say,

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Can you see a pattern?

Let's see if we can use this idea to help us to find the inverse of each of the following matrices.

(1)  $\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix}$

Then by taking the lower right-hand number and writing it in the upper left-hand spot:

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 12 & 7 \\ 5 & 3 \end{pmatrix}$$

Then by taking the lower left-hand number, taking its "opposite" or "additive inverse," and writing that in this same lower left-hand spot:

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \xrightarrow{5 \rightarrow -5} \begin{pmatrix} 12 & 7 \\ -5 & 3 \end{pmatrix}$$

Take additive inverse

Finally, we use a similar procedure on the upper right-hand spot:

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \xrightarrow{7 \rightarrow -7} \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix}$$

Take additive inverse

Suppose, now, we try this same pattern on the matrix of question 1. Will it work? Let's try it, and see! We can suggest the pattern by schematic pictures.

(1) (a)  $\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -7 \\ -5 & 12 \end{pmatrix}$

(b)  $\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 12 & -7 \\ 3 & 12 \end{pmatrix}$

(c)  $\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \xrightarrow{-5 \rightarrow +5} \begin{pmatrix} 3 & -7 \\ +5 & 12 \end{pmatrix}$

Take additive inverse

(d)  $\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \xrightarrow{-7 \rightarrow +7} \begin{pmatrix} 3 & +7 \\ -5 & 12 \end{pmatrix}$

Take additive inverse



Now let's try it out and see if it worked:

$$\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & -7 \\ 5 & 12 \end{pmatrix} = ?$$

We can compute this by matrix multiplication,

$$\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & -7 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the pattern *did* work!

We could describe this pattern, using variables, like this:

The inverse of *any* 2-by-2 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is (if the pattern above always works!!)

$$\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

As a matter of fact, a really ambitious student might try this out at this point, to see if it really does always work. Here is what he would find:

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} &= \begin{pmatrix} AD + B(-C) & A(-B) + BA \\ CD + D(-C) & C(-B) + DA \end{pmatrix} \\ &= \begin{pmatrix} AD - BC & 0 \\ 0 & AD - BC \end{pmatrix} \\ &= (AD - BC) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Obviously, this method works *if and only if*  $AD - BC \neq 0$ . Otherwise, we must *divide* by  $AD - BC$ . Consequently, here is a rule that *always* works, *provided*  $AD - BC \neq 0$ :

If  $AD - BC \neq 0$ , then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} \frac{D}{AD - BC} & \frac{-B}{AD - BC} \\ \frac{-C}{AD - BC} & \frac{A}{AD - BC} \end{pmatrix}$$

[It is possible to show, without too much trouble, that if  $AD - BC = 0$ , then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has no inverse whatsoever. Hence our failure to find one in this case is not merely excusable, it is creditable.]

Of course, in this discussion we have gotten way ahead of your class. Let us now return to your class, who are presumably working on questions 1 through 6.

A second method for solving question 1 goes like this: It is true that matrices do *not* ordinarily satisfy the commutative law for multiplication (CLM), *but in some special cases matrices actually do satisfy CLM*. In particular, every matrix commutes with its own inverse. Hence, if (in the illustrative example)

$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then it must also be true that

$$\begin{pmatrix} 12 & -7 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$


But this last equation tells us that

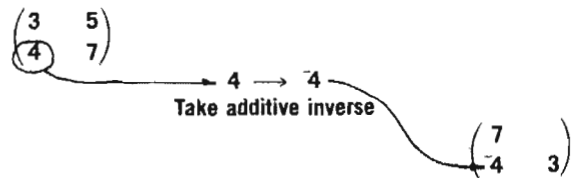
$$\begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}$$

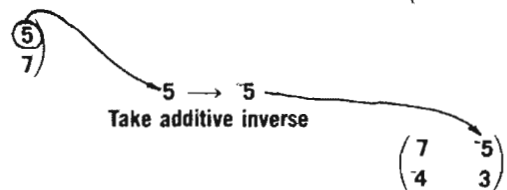
is precisely the inverse we are seeking!

(2)  $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$

(2) (a)  $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$  

(b)  $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$  

(c)  $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$    $\begin{pmatrix} 7 \\ -4 & 3 \end{pmatrix}$

(d)  $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$    $\begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}$

Now! Did it work? Let's try it out and see:

$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix} = ?$$

Multiplying these two matrices together, we get

$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so it *did* work! The matrix

$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$$

has the multiplicative inverse

$$\begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$$

(3)  $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$

(4)  $\begin{pmatrix} 13 & 2 \\ 6 & 1 \end{pmatrix}$

(5)  $\begin{pmatrix} 9 & 10 \\ 8 & 9 \end{pmatrix}$

(6)  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(3)  $\begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$

(4)  $\begin{pmatrix} 1 & -2 \\ -6 & 13 \end{pmatrix}$

(5)  $\begin{pmatrix} 9 & -10 \\ -8 & 9 \end{pmatrix}$

(6)  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Problem 7 is an example of what the Madison Project calls "torpedoing." We have led the students to making a generalization (what J. Richard Suchman would call a "theory"), and the students' theory is a good one as far as it goes. *But it does not go very far!* In fact, we find in problem 7 that the students' method no longer works.

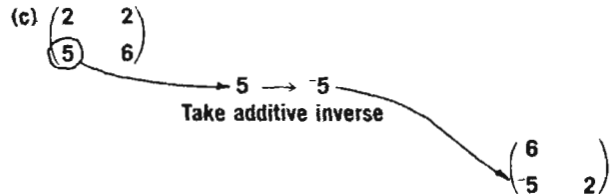
That is not catastrophic. In fact, a result like this is part of the daily experience of scientists, and actually probably of most people, if they are alert enough to notice.

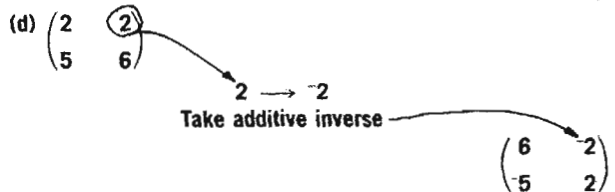
(7)  $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$

(7) Let's try the same "pattern" method we have been using, and see how it works.

(a)  $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$  

(b)  $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$  

(c)  $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$  

(d)  $\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix}$  

Now ... did it work? Let's try:

$$\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 6 & -2 \\ -5 & 2 \end{pmatrix} = ?$$

Multiplying, we get

$$\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 6 & -2 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is *not* what we want. However, it is close! We just got *twice* what we want. Therefore, all we need to do is to take

$$\begin{pmatrix} 6 & -2 \\ -5 & 2 \end{pmatrix}$$

and divide it by 2, to get

$$\begin{pmatrix} 3 & -1 \\ -\frac{5}{2} & 1 \end{pmatrix}.$$

Now let's see if this won't work:

$$\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -\frac{5}{2} & 1 \end{pmatrix} = ?$$

We multiply,

$$\begin{pmatrix} 2 & 2 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -\frac{5}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so the correct inverse is

$$\begin{pmatrix} 3 & -1 \\ -\frac{5}{2} & 1 \end{pmatrix}.$$

Perhaps this is a good example of what Jerrold Zacharias has called "the exploitation of error."

(8)  $\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$

(9)  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(10)  $\begin{pmatrix} 5 & 9 \\ 2 & 4 \end{pmatrix}$

(11)  $\begin{pmatrix} 5 & 6 \\ 8 & 10 \end{pmatrix}$

(12)  $\begin{pmatrix} 25 & 23 \\ 1 & 1 \end{pmatrix}$

[page 168]

(8)  $\begin{pmatrix} 1 & -3 \\ -\frac{1}{2} & 2 \end{pmatrix}$ . This is similar to question 7.

(9)  $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ . This, again, is similar to question 7 (although there are also other ways to solve problem 9).

(10)  $\begin{pmatrix} 2 & -\frac{9}{2} \\ -1 & \frac{5}{2} \end{pmatrix}$ . Similar to question 7.

(11)  $\begin{pmatrix} 5 & -3 \\ -4 & -\frac{5}{2} \end{pmatrix}$ . Similar to question 7.

(12)  $\begin{pmatrix} \frac{1}{2} & -\frac{23}{2} \\ -\frac{1}{2} & \frac{25}{2} \end{pmatrix}$ . Similar to question 7.

(13)  $\begin{pmatrix} 8 & 5 \\ 1 & 1 \end{pmatrix}$

(14)  $\begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$

(15)  $\begin{pmatrix} 7 & 5 \\ 1 & 1 \end{pmatrix}$

(16)  $\begin{pmatrix} 7 & 4 \\ 1 & 1 \end{pmatrix}$

(17)  $\begin{pmatrix} A & B \\ 1 & 1 \end{pmatrix}$

Some of these problems seem to be harder than others. If we can see which problems are "easier" and which are "harder," that may give us a clue as to how to proceed.

If any of your answers thusfar have been wrong, those problems deserve special attention! Professor Jerrold Zacharias, a physicist at Massachusetts Institute of Technology, has suggested that the "exploitation of error" is a powerful tool in scientific research. What can we learn from looking carefully at the problems that were wrong? How were they different from those we got right? In what way were the wrong answers "wrong"? Were they completely wrong or almost correct?

If you want, make up some matrices yourself and try to find their inverses. If you have trouble, see what you can learn by "exploiting error."

Some people claim that the idea of determinants, from Chapter 46, can be helpful to us. If you wish, see if determinants really can be helpful.

(13)  $\begin{pmatrix} \frac{1}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ . This is somewhat similar to question 7, except that in problem 13 it is necessary to divide by 3, instead of 2.

Can you find a method to tell in advance whether you will need to divide or not, and if so, by what number? (It is not necessary, but at this point you could go back to the remarks following the answer to question 1 and see what ideas they may suggest.)

(14)  $\begin{pmatrix} \frac{1}{4} & -\frac{5}{4} \\ -\frac{1}{4} & \frac{2}{4} \end{pmatrix}$ . Similar to question 13, except that in problem 14 you need to divide by 4, instead of 2 or 3.

(15)  $\begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & \frac{2}{2} \end{pmatrix}$ . Similar to question 7.

(16)  $\begin{pmatrix} \frac{1}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ . Similar to question 13.

(17) Let's try the method that was suggested by the remarks following the answer to question 1.

If  $AD - BC \neq 0$ , then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} \frac{D}{AD - BC} & \frac{-B}{AD - BC} \\ \frac{-C}{AD - BC} & \frac{A}{AD - BC} \end{pmatrix}$$

To apply this method to question 17, we use UV, as follows:

$$\begin{aligned} UV: A &\rightarrow A \\ &B \rightarrow B \\ &1 \rightarrow C \\ &1 \rightarrow D \end{aligned}$$

Then  $AD - BC = A - B$ , and the method will not work if  $A - B = 0$ . (You can easily show that in this case there is no inverse whatsoever.)

However, if  $A - B \neq 0$ , the inverse should be

$$\begin{pmatrix} \frac{1}{A - B} & \frac{-B}{A - B} \\ \frac{-1}{A - B} & \frac{A}{A - B} \end{pmatrix}$$

Let's try it out and see if it works:

$$\begin{pmatrix} A & B \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{A - B} & \frac{-B}{A - B} \\ \frac{-1}{A - B} & \frac{A}{A - B} \end{pmatrix} = ?$$

Multiplying out, we get

$$\begin{aligned} \begin{pmatrix} A & B \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{A-B} & \frac{\circ B}{A-B} \\ \frac{-1}{A-B} & \frac{A}{A-B} \end{pmatrix} \\ = \begin{pmatrix} \frac{A-B}{A-B} & \frac{A \times (B + \circ B)}{A-B} \\ \frac{1-1}{A-B} & \frac{\circ B + A}{A-B} \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

(18) Can you find the inverse for any matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

(19) When can you find the inverse for the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} ?$$

(18) It is not possible to find the multiplicative inverse for every 2-by-2 matrix. If  $AD - BC = 0$ , then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

does not have any multiplicative inverse.

(19) At this point we can state the complete rule:

Case 1. If  $AD - BC = 0$ , the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

does not have any multiplicative inverse.

Case 2. If  $AD - BC \neq 0$ , then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} \frac{D}{AD-BC} & \frac{\circ B}{AD-BC} \\ \frac{\circ C}{AD-BC} & \frac{A}{AD-BC} \end{pmatrix}.$$

Hopefully, thinking carefully about the questions in this chapter should have led your students to *conjecture* this result (at least for case 2, which is the one we presently care about).

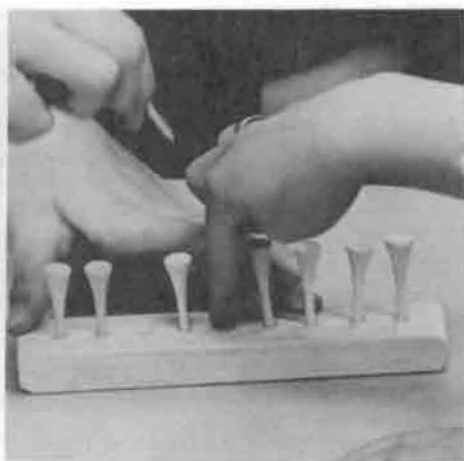
But once you *guess* this result, it is perfectly easy to try out your guess and see whether or not it really does work:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} \frac{D}{AD-BC} & \frac{\circ B}{AD-BC} \\ \frac{\circ C}{AD-BC} & \frac{A}{AD-BC} \end{pmatrix} = ?$$

We multiply out:

$$\begin{aligned}
 \begin{pmatrix} A & B \\ C & D \end{pmatrix} &\times \begin{pmatrix} D & -B \\ -C & A \\ \hline AD - BC & AD - BC \end{pmatrix} \\
 &= \begin{pmatrix} AD + B(-C) & A(-B) + BA \\ AD - BC & AD - BC \\ \hline CD + D(-C) & C(-B) + DA \\ AD - BC & AD - BC \end{pmatrix} \\
 &= \begin{pmatrix} AD - BC & A(-B + B) \\ AD - BC & AD - BC \\ \hline C(-B + D) & AD - BC \\ AD - BC & AD - BC \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

Voilà.



## Appendix A

### *Suggestions for Further Reading— An Annotated Bibliography*

The study of mathematics is more than life long. No one achieves, within his lifetime, a complete knowledge of all mathematics. (To make matters worse, the same can be said of education, psychology, history, or any of the other subjects with which a teacher deals.)

I am sure that most readers will have questions that are not answered in this text. The following books may help.

#### **PEDAGOGY AND PHILOSOPHY—I**

Perhaps the most immediate questions will relate to what *Explorations* (and the "new curriculum" projects in general) are trying to accomplish for children. There is, fortunately, an excellent essay which may answer many of these questions:

- (1) Schwab, J. J., "The Teaching of Science as Inquiry," in J. J. Schwab and Paul F. Brandwein, *The Teaching of Science* (Harvard Univ. Press, Cambridge, Mass., 1964).

Also of considerable interest are:

- (2) Bruner, Jerome S., *Toward a Theory of Instruction* (Harvard Univ. Press, Cambridge, Mass., 1966).
- (3) Holt, John, *How Children Fail* (Pitman, New York, 1964).
- (4) Mearns, Hughes, *Creative Power* (Dover, New York, 1958).

#### **PEDAGOGY AND PHILOSOPHY—II**

- (5) Ashton-Warner, Sylvia, *Teacher* (Simon and Schuster, New York, 1963).
- (6) Avers, Paul W., "A Unit in High School Geometry Without the Textbook," *The Mathematics Teacher*, Vol. LVII, No. 3 (March, 1964), pp. 139-142.
- (7) Berne, Eric, *Games People Play* (Grove Press, New York, 1964).
- (8) Boole, Mary Everest, *The Preparation of the Child for Science* (Oxford Univ. Press, New York, 1904).
- (9) Boullé, Pierre, *The Test* (Popular Library, New York, 1960).

- (10) Brecher, Ruth, and Edward Brecher, "Gifted Children Need Freedom to Learn," *Parents' Magazine* (June, 1962), pp. 44 ff.
- (11) Bruner, Jerome S., *The Process of Education* (Harvard Univ. Press, Cambridge, Mass., 1963).
- (12) —, "The Course of Cognitive Growth," *American Psychologist*, Vol. 19, No. 1 (1964).
- (13) Cantor, Nathaniel, *The Dynamics of Learning* (Henry Stewart, New York, 1961).
- (14) Clarkson, David, "Taxicab Geometry, Rabbits, and Pascal's Triangle—Discoveries in a Sixth-Grade Classroom," *The Arithmetic Teacher*, Vol. IX, No. 6 (October, 1962), pp. 308-313.
- (15) Cleary, J. Robert, *A Study of Test Performance in Two Madison Project Schools and One Control School* (Madison Project, Webster Groves, Mo., 1965).
- (16) Cohen, Donald, "A Lesson on Absolute Value," *The Arithmetic Teacher*, Vol. 11, No. 8 (December, 1964), pp. 561 and 562.
- (17) Courtis, Stuart A., "Our Choice: Revolution or Destruction," *Educational Leadership*, Vol. 20, No. 8 (May, 1963), pp. 520-522.
- (18) Davis, Robert B., "Solving Problems and Constructing Systems—Quadratic Equations and Vectors," Report of an Orientation Conference for SMSG Experimental Centers, Chicago, Illinois (September 19, 1959), pp. 97-101.
- (19) —, *The Madison Project: A Brief Introduction to Materials and Activities* (Madison Project, Webster Groves, Mo., 1962).
- (20) —, *Notes on the Film: A Lesson With Second Graders*, Booklet to accompany film (Madison Project, Webster Groves, Mo., 1962).
- (21) —, *Notes on the Film: First Lesson*, Booklet to accompany film (Madison Project, Webster Groves, Mo., 1962).
- (22) —, *Notes on the Film: Matrices*, Booklet to accompany film (Madison Project, Webster Groves, Mo., 1962).
- (23) —, "The Evolution of School Mathematics," *Journal of Research in Science Teaching*, Vol. 1 (1963), pp. 260-264.
- (24) —, "Report on Madison Project Activities, September, 1962—November, 1963," Report submitted to the National Science Foundation (December, 1963).



- (25) ———, *A Modern Mathematics Program as It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere (The Madison Project)*, 1963 (Report submitted to the Commissioner of Education, U.S. Dept. of Health, Education, and Welfare).
- (26) ———, *A Modern Mathematics Program as It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere (The Madison Project)*, 1965 (Report submitted to the Commissioner of Education, U.S. Dept. of Health, Education, and Welfare).
- (27) ———, *Experimental Course Report/Grade Nine* (Report #1, June, 1964), (Madison Project, Webster Groves, Mo.).
- (28) ———, "The Madison Project's Approach to a Theory of Instruction," *Journal of Research in Science Teaching*, Vol. 2 (1964), pp 146-162.
- (29) ———, "Recent Activities of the Madison Project," *American Mathematical Monthly* (January, 1965).
- (30) ———, "What Do We Mean by Discovery?" Paper written for presentation at the meeting on "discovery" learning, Social Science Research Council, January 28-29, 1965.
- (31) ———, "Math Takes a New Path," *The PTA Magazine*, Vol. LVII, No. 6 (February, 1963), pp. 8-11.
- (32) ———, "Some Remarks on 'Learning By Discovery'," (Madison Project, Webster Groves, Mo., 1965).
- (33) de Chardin, Teilhard, *The Phenomenon of Man* (Harper Torchbooks, New York, 1961).
- (34) Dienes, Z. P., *Power of Mathematics*, pp. 17-25 (Hutchinson Educational Ltd., London, 1964).
- (35) Feynman, R. P., R. B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1963).
- (36) Gallagher, James J., *Research Trends and Needs in Educating the Gifted, A Critique* (U.S. Department of Health, Education, and Welfare, Office of Education, Superintendent of Documents Catalog No. FS S. 235:35036, 1964).
- (37) Gibb, Glenadine, Phillip Jones, and Charlotte Junge, *The Growth of Mathematical Ideas, Grades K-12* (24th yearbook of the National Council of Teachers of Mathematics).
- (38) Goodman, Paul, *Compulsory Mis-education*, pp. 53, 54, and 81 (Horizon Press, New York, 1964).
- (39) Gross, Ronald, "Two-Year-Olds are Very Smart," *The New York Times*, Magazine Section (September 6, 1964), pp. 10-11.
- (40) ———, "What We Don't Know May Help Us," *New York Herald Tribune*, Book Week (February 7, 1965), pp. 5 and 15.
- (41) Hawkins, David, "On Living in Trees" (Karl Muenzinger Memorial Lecture, Univ. of Colorado, Boulder, Colo., 1964).
- (42) Henry, Jules, *Culture Against Man* (Random House, New York, 1963).
- (43) Huxley, Aldous, "The Education of an Amphibian," in *Tomorrow and Tomorrow and Tomorrow* (Signet Books, New York, 1964).
- (44) ———, "Education on the Nonverbal Level" in Alfred de Gratia and David A. Sohn, *Revolution in Teaching: New Theory, Technology, and Curricula* (Bantam Books, New York, 1964).
- (45) Kaufman, Bel, *Up the Down Staircase* (Prentice-Hall, Englewood Cliffs, N.J., 1964).
- (46) Kelley, Earl C., *The Workshop Way of Learning* (Harper, New York, 1951).
- (47) Kemp, C. Gratton, "Comparison of Manifest Needs of Open and Closed Minds," *Journal of Research in Science Teaching*, Vol. 2, Issue 2 (June, 1964).
- (48) Kersh, Bert Y., "Learning by Discovery: What is Learned?," *The Arithmetic Teacher*, Vol. 11 (1964), pp. 226-232.
- (49) Matthews, Warren N., "Letter to the Editor," *The Mathematics Teacher*, Vol. LVII, No. 3 (March, 1965), p. 231.
- (50) McClelland, David C., *The Achieving Society*
- (51) MINNE-MAST Project, *An Overview of the MINNE-MAST Mathematics Curriculum, Kindergarten-3rd Grade* (Univ. of Minnesota, Minneapolis, Minn., April, 1965).
- (52) Murphy, Lois B., et al., *The Widening World of Childhood* (Basic Books, New York, 1962).
- (53) Neill, A. S., *Summerhill* (Hart, New York, 1960).
- (54) Page, David, "Well-Adjusted Trapezoids," *Updating Mathematics*, Vol. 4, No. 8, Section 11 (Croft Educational Services, New London, Conn., 1962).
- (55) Pincus, Morris, "An Adventure in Discovery," *The Arithmetic Teacher*, Vol. 11, No. 1 (January, 1964), pp. 28-29.
- (56) Polya, George, *Mathematics and Plausible Reasoning*, 2 volumes (Princeton Univ. Press, Princeton, N.J., 1954).
- (57) ———, *How To Solve It* (Doubleday, New York, 1957).
- (58) ———, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, 2 volumes (Wiley, New York, 1965).
- (59) Reik, Theodor, *Listening With the Third Ear* (Farrar, New York, 1948).
- (60) Rockcastle, Verne N., et al., *Piaget Re-discovered*. A report of the Conference on Cognitive Studies and Curriculum Development, March, 1964 (School of Education, Cornell Univ., Ithaca, N.Y.).
- (61) Rogers, Carl R., *Client-Centered Therapy* (Houghton Mifflin, Boston, Mass., 1959).
- (62) ———, *On Becoming a Person* (Houghton Mifflin, Boston, Mass., 1961).
- (63) Report of the Cambridge Conference on School Mathematics, *Goals for School Mathematics* (Houghton Mifflin, Boston, Mass., 1963).
- (64) Sanders, W. J., "The Use of Models in Mathematics Instruction," *The Arithmetic Teacher*, Vol. 11, No. 3 (March, 1964), pp. 157-165.

- (65) Sawyer, W. Warwick, *Vision in Elementary Mathematics* (Penguin Books, Baltimore, Md., 1964).
- (66) Skinner, B. F., "Why Teachers Fail," *Saturday Review* (October 16, 1965), pp. 80 ff.
- (67) Smedslund, Jan, "Internal Necessity and Contradiction in Children's Thinking," *Journal of Research in Science Teaching*, Vol. 2, No. 3 (September, 1964), pp. 220-221.
- (68) Snyder, Henry D., "An Impromptu Discovery Lesson in Algebra," *The Mathematics Teacher* (October, 1964), pp. 415-416.
- (69) Tolman, Edward Chace, "Cognitive Maps in Rats and Men," *Behavior and Psychological Man*, Chapter 19 (Univ. of California Press, Berkeley, Calif., 1958).
- (70) Torrance, E. Paul, "Creativity" Research Pamphlet Series, *What Research Says to the Teacher*, #28, April, 1963 (Department of Classroom Teachers, American Educational Research Association of the National Education Association).
- (71) Whitehead, Alfred North, *Aims of Education* (Mentor Books, New York, 1929).

#### MATHEMATICS

For most readers, the critical need will probably be for further help in the study of mathematics itself. Here are some fairly basic books:

- (72) Dupree, Daniel E., and Frank L. Harmon, *Modern College Algebra* (Prentice-Hall, Englewood Cliffs, N.J., 1965). This is fairly similar to Professor Fine's book (reference 74), but is possibly somewhat easier. Topics match closely with *Explorations*: logic, sets, functions, graphs, complex numbers, exponents, open sentences, inequalities, simultaneous equations, determinants, matrices, the exponential function, the binomial theorem, etc.
- (73) Hummel, James A., *Vector Geometry* (Addison-Wesley, Reading, Mass., 1965). Despite its apparently specialized title, this volume provides a good general mathematical background for much of the material dealt with in *Explorations*.
- (74) Fine, Nathan J., *Introduction to Modern Mathematics* (Rand McNally, Chicago, Ill., 1965). This is a very good book, carefully written by a very good mathematician. It deals with many of the same topics found in *Explorations*: logic, functions, sets, graphs, axioms, matrices, simultaneous equations, probability, isomorphism, transformations, etc. It does not require a strong previous background in mathematics—it starts from a high school level.
- Some teachers may feel that what they really need is a modern approach to ninth-grade algebra. Several good books are available, for example:
- (75) Brumfiel, Charles, Robert Eicholz, and Merrill Shanks, *Algebra I* (Addison-Wesley, Reading, Mass., 1961).
- In the general area of relatively simple books dealing with the ideas which we have encountered in *Explorations* there are many that might be recommended. Here are a few:
- (76) Allendoerfer, Carl B., and Cletus O. Oakley, *Principles of Mathematics* (McGraw-Hill, New York, 1963). Topics include matrices, logic, sets, axioms, isomorphism, simultaneous equations, functions, graphs, complex numbers, going even into ideas of calculus. This book has been popular for a long time (it was originally published in 1955), and is usually considered quite readable.
- (76a) Byrne, J. Richard, *Modern Elementary Mathematics* (McGraw Hill, New York, 1966).
- (77) Fletcher, T. J., et al., *Some Lessons in Mathematics. A Handbook on the Teaching of "Modern" Mathematics* (Cambridge Univ. Press, New York, 1964). This is a very valuable book. Teachers in the United States will be intrigued by the point of view of their British colleagues, as revealed in this book and in others that may be forthcoming from this same group of authors. Topics include geometrical mappings or transformations, graphs, matrices, vectors, sets, logic, binary numerals, and industrial applications.
- (78) Montague, Harriet, and Mabel Montgomery, *The Significance of Mathematics* (Charles Merrill Books, Columbus, Ohio, 1963). Topics include matrices, sets, logic, axioms, history, and statistics.
- (79) Sanders, Paul, *Elementary Mathematics—A Logical Approach* (International Textbook Co., Scranton, Pa., 1963).
- (79a) Howard, Charles F., and Enoch Dumas, *Teaching Contemporary Mathematics in the Elementary School* (Harper and Row, New York, 1966).
- Of somewhat different interest are these books:
- (80) Exner, Robert M., and Myron F. Roszkopf, *Logic in Elementary Mathematics* (McGraw-Hill, New York, 1959). This is a valuable book for those who wish to pursue logic further than we have been able to go with it in *Explorations*.
- (81) Jones, Burton W., *Elementary Concepts of Mathematics* (Macmillan, New York, 1947). This is a particularly good reference for anyone who is puzzled by the "symmetry," "reflection," and "transformation" ideas in *Explorations*, and who wishes to understand these ideas better. (For this same topic, consult the film "Reflection," produced by David Roseveare of the British Broadcasting Corporation (BBC), and pages 16-23 of the BBC pamphlet *Middle School Mathematics, Autumn, 1964*. These are not readily available, but they do exist; contact David Roseveare, Kensington House, Room 505, London, England.)
- (82) Kemeny, John, Laurie Snell, and Gerald L. Thompson, *Introduction to Finite Mathematics* (Prentice-Hall, Englewood Cliffs, N.J., 1956). This book contains some unusual topics, particularly in relation to matrices and to trees.

- (83) Mosteller, Frederick, R. E. K. Rourke, and G. B. Thomas, Jr., *Probability: A First Course* (Addison-Wesley, Reading, Mass., 1961). An excellent book for those who wish to pursue the ideas of statistics and probability.

For those who have an adequate knowledge of the contents of the books listed above, and who want to go on to more advanced mathematics, the field is wide open. I shall list three books, out of several hundred possible choices:

- (84) Birkhoff, Garrett, and Saunders MacLane, *A Survey of Modern Algebra* (MacMillan, New York, 1944). This book has long been, deservedly, the standard reference book in algebra.
- (85) Britton, Jack R., R. Ben Krieger, and Leon Rutland, *University Mathematics*, Vol. 1 (Freeman, San Francisco, Calif., 1965). An introduction to calculus, from a point of view that appears to be consonant with that of *Explorations*.
- (86) Eves, Howard, and Carroll V. Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics* (Holt, New York, 1958). An excellent book.

Also of interest:

- (87) Beberman, Max, and Herbert E. Vaughan, *High School Mathematics* (D. C. Heath, Boston, Mass., 1964).
- (88) Crouch, Ralph, and David Beckman, *Linear Algebra* (Scott Foresman, Chicago, Ill., 1965).
- (89) Curry, Haskell, *Foundations of Mathematical Logic*, (McGraw-Hill, New York, 1963).
- (90) Levi, Howard, *Elements of Algebra* (Chelsea, New York, 1960).
- (91) Lister, Frederick M., Sheldon T. Rio, and Walter J. Sanders, *Freshman Mathematics for University Students* (Prentice-Hall, Englewood Cliffs, N.J., 1964).
- (92) Maria, May Hickey, *The Structure of Arithmetic and Algebra* (Wiley, New York, 1958).
- (93) Mendelson, Elliott, *Introduction to Mathematical Logic* (Van Nostrand, Princeton, N.J., 1964).
- (94) Moise, E. E., *Elementary Geometry from an Advanced Standpoint* (Addison-Wesley, Reading, Mass., 1963).
- (95) ———, *The Number Systems of Elementary Mathematics* (Addison-Wesley, Reading, Mass., 1966).
- (96) Newman, J. R., and E. Nagel, *Gödel's Proof* (New York Univ. Press, New York, 1960).
- (97) Ohmer, M. M., C. V. Aucoin, and M. J. Cortez, *Elementary Contemporary Mathematics* (Blaisdell, New York, 1964).
- (98) Suppes, P., and S. Hill, *First Course in Mathematical Logic* (Blaisdell, New York, 1964).
- (99) Swain, Robert L., "Logic: For Teacher, For Pupil" in *Enrichment Mathematics for the Grades*. 27th Yearbook of the National Council of Teachers of Mathematics.

## RESEARCH

Those interested in on-going research efforts related to the kind of educational experiences discussed in *Explorations* may want to read the following, in addition to the books and articles listed earlier:

- (100) Arons, A. B., and A. M. Bork, Eds., *Science and Ideas* (Prentice Hall, Englewood Cliffs, N.J., 1964).
- (101) Ausubel, David P., *Implications of Preadolescent and Early Adolescent Cognitive Development for Secondary School Teaching* (mimeographed), (Bureau of Educational Research, Univ. of Illinois, Urbana, Ill.).
- (102) Barnes, Fred P., *Research for the Practitioner in Education* (Department of Elementary School Principals, N.E.A., Washington, D.C., 1964).
- (103) Beberman, Max, *An Emerging Program of Secondary School Mathematics* (Harvard Univ. Press, Cambridge, Mass., 1958).
- (104) Boring, Edwin G., *History, Psychology, and Science*, Watson and Campbell, Eds. (Wiley, New York, 1963).
- (105) Brown, Roger, *Social Psychology* (The Free Press, New York, 1965).
- (106) Callahan, Raymond E., *Education and The Cult of Efficiency* (Univ. of Chicago Press, Chicago, Ill., 1962).
- (107) Coddington, Earl A., "Scholastic Aptitude Tests in Mathematics," *The American Mathematical Monthly* (August-September, 1963), pp 750-755.
- (108) Combs, Arthur W., et al., *Perceiving, Behaving, Becoming* (Assoc. Supervision and Curriculum Development, N.E.A., Washington, D.C.).
- (109) Committee of the College and University Examiners, *Taxonomy of Educational Objectives, I: Cognitive Domain*, Benjamin S. Bloom, Ed. (Longmans, Green and Co., New York, 1956).
- (110) Day, Robert C., and Robert L. Hamblin, "Some Effects of Close and Punitive Styles of Supervision," *American Journal of Sociology*, Vol. LXIX, No. 5 (March, 1964), pp. 499-510.
- (111) de Charms, Richard, Virginia Carpenter, and Aharon Kuperman, "The 'Origin-Pawn' Variable in Person Perception," *Sociometry* (to appear).
- (112) Dexter, Lewis A., *The Tyranny of Schooling* (Basic Books, New York, 1964).
- (113) Erikson, Erik H., "Identity and the Life Cycle," *Psychological Issues*, Vol. 1, No. 1 (1959), monograph 1.
- (114) Flavell, John H., *The Developmental Psychology of Jean Piaget* (Van Nostrand, Princeton, N.J., 1963).
- (115) Gage, N. L., *Handbook of Research on Teaching* (Rand McNally, Chicago, Ill., 1963).
- (116) Gagne, Robert M., *The Conditions of Learning* (Holt, New York, 1965).
- (117) Gleason, Andrew, et al., *Goals for School Mathematics*, The Report of the Cambridge Conference on School Mathematics (Houghton Mifflin, Boston, Mass., 1963).

- (118) Goutard, Madeleine, *Mathematics and Children* (Education Explorers Ltd., Reading, England, 1964).
- (119) Hadamard, Jacques, *The Psychology of Invention in the Mathematical Field* (Dover, New York, 1945).
- (120) Hardin, Blanche C., "Math—Not New," *The Arithmetic Teacher*, Vol. 12, No. 4 (April, 1965), p. 252.
- (121) Heath, Robert, Ed., *New Curricula* (Harper, New York, 1964).
- (122) Hilgard, Ernest R., Ed., *Theories of Learning and Instruction* (Univ. of Chicago Press, Chicago, Ill., 1964).
- (123) Holt, John, "How to Help Babies Learn—Without Teaching Them," *Redbook*, Vol. 126, No. 1 (November, 1965), pp. 55 ff.
- (124) Hoyle, Fred, *The Black Cloud* (Harper, New York, 1958).
- (125) Hunt, J. McV., "Revisiting Montessori," in Montessori and Hunt, *The Montessori Method* (Schoken, New York, 1964).
- (126) Inbody, Donald, "Helping Parents Understand New Mathematics Programs," *The Arithmetic Teacher*, Vol. 11, No. 8 (December, 1964), pp. 530-537.
- (127) Karplus, Robert, "The Science Curriculum Improvement Study," *Journal of Research in Science Teaching*, Vol. 2, Issue 4 (December, 1964), pp. 293-303.
- (128) Krathwohl, David R., et al., *Taxonomy of Educational Objectives, II: Affective Domain* (David McKay, New York, 1964).
- (129) Lewin, Kurt, *Field Theory in Social Science*, Dorwin Cartwright, Ed. (Harper, New York, 1951).
- (130) Manheim, Jerome H., and Sylvia R. Manheim, "Connectionism and the Teaching Machine," *The Mathematics Teacher*, Vol. LVIII, No. 3 (March, 1965), pp. 200-204.
- (131) May, Kenneth O., *Programed Learning and Mathematical Education* (Committee on Educational Media, Mathematical Association of America, 1965).
- (132) Moise, E. E., "Activity and Motivation in Mathematics," *American Mathematical Monthly*, Vol. 72, No. 4 (April, 1965), pp. 407-412.
- (133) Pfeiffer, John, "When Man First Stood Up," *New York Times*, Magazine Section (April 11, 1965). The date was omitted from this printing of the Magazine Section, but April 11 appears to be the actual date.
- (134) Polanyi, Michael, *Personal Knowledge* (Univ. of Chicago Press, Chicago, Ill., 1958).
- (135) Rosenbloom, Paul C., Ed., *Modern Viewpoints in the Curriculum* (McGraw-Hill, New York, 1964).
- (136) Sealey, L. G. W., *The Creative Use of Mathematics in the Junior School* (Blackwell, Oxford, England, 1961).
- (137) Smith, Lewis B., "Pegboard Geometry," *The Arithmetic Teacher*, Vol. 12, No. 4 (April, 1965), pp. 271-274.
- (138) Tagiuri, Renato, and Luigi Petruccio, Eds., *Person Perception and Interpersonal Behavior* (Stanford Univ. Press, Stanford, Calif., 1958).
- (139) Teevan, Richard, and Robert Birney, Eds., *Theories of Motivation in Personality and Social Psychology* (Van Nostrand, Princeton, N.J., 1964).
- (140) Thelen, Herbert A., "Educational Dynamics: Theory and Research," *The Journal of Social Issues*, Vol. 6, No. 2 (1950).
- (141) Vygotsky, Lev Semenovich, *Thought and Language*, Hanfmann and Vakar, Trans. (M.I.T. Press, Cambridge, Mass., 1962).
- (142) Wann, T. W., Ed., *Behaviorism and Phenomenology* (Univ. Chicago Press, Chicago, Ill., 1964).
- (143) Weil, Andrew, "Harvard's Bruner and His Yeasty Ideas," *Harper's Magazine* (December, 1964), pp. 81-89.
- (144) Werner, Heinz, and Bernard Kaplan, *Symbol Formation* (Wiley, New York, 1963).
- (145) Wertheimer, Max, *Productive Thinking* (Harper, New York, 1959).
- (146) Wittrock, M. C., "The Learning by Discovery Hypothesis," mimeographed (U.C.L.A., Los Angeles, Calif., December, 1964).

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- (147) Ardrey, Robert, *African Genesis* (Dell, New York, 1963).
- (148) Barck, O. T., Jr., and H. T. Lefler, *Colonial America* (Macmillan, New York, 1958).
- (149) Bell, Eric Temple, *Men of Mathematics* (Simon and Schuster, New York, 1961).
- (150) Clark, G. N., *The Seventeenth Century* (Oxford Univ. Press, New York, 1953).
- (150a) de Heinzelin, Jean, "Ishango," *Scientific American* (June, 1962), pp. 105-116.
- (151) Eves, Howard, *An Introduction to the History of Mathematics*, rev. ed. (Holt, New York, 1964).
- (152) Haskins, Charles Homer, *The Rise of Universities* (Cornell Univ. Press, Ithaca, N.Y., 1957).
- (153) Hofstadter, Richard, *Academic Freedom in the Age of the College* (Columbia Univ. Press, New York, 1955).
- (154) Kramer, Samuel Noah, *History Begins at Sumer* (Doubleday-Anchor, Garden City, N.Y., 1959).
- (155) Leason, James, *The Plague and the Fire* (Avon, New York, 1961).
- (156) Lloyd, Daniel B., "Recent Evidences of Primeval Mathematics," *The Mathematics Teacher*, Vol. LVII, No. 8 (December, 1965), pp. 720-723.
- (157) Rowe, Anne, "Psychology and the Evolution of Man," *Harvard Educational Review*, Vol. 36, No. 2 (Spring, 1966) pp. 139-154.
- (158) Newman, James R., Ed., *The World of Mathematics*, 4 volumes (Simon and Schuster, New York, 1956). These four volumes deserve to be in your classroom.
- (159) Orlich, Donald C., "The Dawn of Scientific Epistemology: 1564-1964," *Journal of Research in Science Teaching*, Vol. 2, Issue 2 (June, 1964), pp. 95-99.
- (160) Tax, Sol, Ed., *Evolution After Darwin*, 3 volumes (Univ. of Chicago Press, Chicago, Ill., 1960).

- (161) Woodward, G. W. O., *A Short History of 16th Century England: 1485-1603* (Mentor Books, New American Library, New York, 1963).
- (162) Wright, Louis B., *The Cultural Life of the American Colonies, 1607-1763* (Harper, New York, 1957).

### NEW CURRICULUM PROJECTS

Teachers usually also want to know about the various "new curriculum" projects in mathematics and in science. Many of these are temporary (though this does not mean they are less valuable), so a definitive listing is not feasible. There are, however, excellent sources of up-to-date information concerning these various projects, and similar matters. Six of the best are:

- (163) *The Arithmetic Teacher* (National Council of Teachers of Mathematics, Washington, D.C.).
- (164) The Commission on Current Curriculum Developments of the Association for Supervision and Curriculum Development (N.E.A., Washington, D.C.). Several reports of this Commission have appeared, and others presumably will in the future.
- (165) Information Clearinghouse on New Science and Mathematics Curricula (Science Teaching Center, University of Maryland, College Park, Md.). Their *Third Report*, compiled by J. David Lockhard and dated March, 1965, has just been released. Single copies are available free of charge.
- (166) *Journal of Research in Science Teaching* (National Association for Research in Science Teaching, New York). Despite its title, this excellent journal deals with both science and with mathematics.
- (167) *The Mathematics Teacher* (National Council of Teachers of Mathematics, Washington, D.C.).
- (168) *Mathematics Teaching* (Association of Teachers of Mathematics, Kent, England). An important

group of educators now active in England publish through this organization.

### CLASSROOM MATERIALS

Teachers may also be interested in books and other materials that are suitable for use in the classroom, for various grade levels from kindergarten to college.

I shall not list basic textbooks, for such a list would be far too long. Here, however, are some less well-known texts that may be useful:

- (169) Churchill, Eileen, *Counting and Measuring* (Routledge and Kegan Paul, London, England).
- (170) Davis, Robert B., *Discovery in Mathematics* (Addison-Wesley, Palo Alto, California, 1964). This book, which we have referred to earlier, is a companion volume to *Explorations*.
- (171) Faber, Norman, Phyllis Gross, and Robert Hansen, *Mathematics and Living Things* (School Mathematics Study Group, Stanford Univ., Stanford, Calif., 1964).
- (172) Page, David A., *Number Lines, Functions, and Fundamental Topics* (Macmillan, New York, 1964).

Useful materials are available from the following sources:

Cuisenaire rods, and other materials, are available from the Cuisenaire Company of America, Inc., 9 Elm Ave., Mt. Vernon, New York 10550.

Games and puzzles (such as the Tower of Hanoi), many of which pose interesting mathematical questions, are available from World Wide Games, R.R. 1, Radnor Rd., Delaware, Ohio 43015.

Z. P. Dienes, of the University of Sherbrooke, Canada, has many interesting pieces of physical apparatus for use in the classroom.

Some unusually important books have just appeared or just come to my attention. Probably the best explanation of how to operate an elementary school mathematics program on the basis of student activity, using physical materials, is given in the various new publications of the Nuffield Mathematics Project. These bear titles such as *I Do—And I Understand* (which accompanies a film with this same title), *Desk Calculators*, *Beginnings*, *Computation and Structure*, *Shape and Size*, and *Pictorial Representation*. These books are available from the Nuffield Foundation Mathematics Teaching Project, 12 Upper Belgrave Street, London, S.W. 1, England.

In addition, for those who are curious and confused about "what's going on," it may be valuable to see the present curriculum revision movement in an appropriate historical perspective by reading the highly relevant book *The Transformation of the School*, by Lawrence A. Cremin (Random House, 1961).

## Appendix B

# Madison Project Films Relevant to Explorations

Some of these films are readily available; others soon will be. Running times are only approximate. For information, write to:

The Madison Project  
Webster College  
St. Louis, Missouri 63119

*Notice that nearly all Madison Project films show actual classroom lessons. They are intended to be viewed by teachers, not by children. They attempt to aid the teacher in planning her next lesson.*

Classroom Social Organization (Small groups vs. large groups, individualizing instruction, etc.)

Some of the films listed under this heading are controversial, for several different reasons. Please do not judge them all on the basis of viewing only one or two. They are by no means all alike.

Large group instruction: "Graphing a Parabola" (6th graders; running time 22 minutes); "Open Sentences and the Number Line" (2nd graders; 9 minutes); "Guessing Functions" (6th and 7th graders; 16 minutes); "Experience with Fractions, Lesson 2" (2nd graders; 30 minutes).

Small group and individualized instruction: "Using Geoboards with Second Graders" (2nd graders; 26 minutes); "Small-group Instruction in Mathematics" (6th graders; 27 minutes); "Small-group Instruction: Signed Numbers, Rational Approximations, and Motion Geometry" (6th graders; 46 minutes); "Small-group Instruction: Committee Report on Signed Numbers" (6th graders; 13 minutes); "Small-group Instruction: Committee Report on Rational Approximations" (6th graders; 22 minutes); "Creative Learning Experiences" (last section, with 8th graders).

Chapter 1 "First Lesson" (students from grades 3-7; running time 1 hour)  
"A More Formal Approach to Variables" (4th graders; 30 minutes)

"A Lesson with Second Graders"\* (2nd graders; about 30 minutes)  
Chapter 2 "First Lesson"  
"A Lesson with Second Graders"\*  
Chapter 3 "Second Lesson" (grades 3-7; 1 hour)  
"Experience with Linear Graphing" (grade 4; 25 minutes)  
Chapter 4 "First Lesson"  
"A Lesson with Second Graders"  
Chapters 5-6 "Introduction to Postman Stories" (grades 4-6; 13 minutes)  
"Small-Group Instruction: Signed Numbers, Rational Approximations, and Motion Geometry" (grade 6; 46 minutes)  
"Small-Group Instruction: Committee Report on Signed Numbers" (grade 6; 13 minutes)  
"Postman Stories" (grade 7; 33 minutes)  
"Three Approaches to Signed Numbers" (grade 9; 65 minutes)  
Chapter 7 "Education Report: The New Math" (30 minutes)  
Chapter 8 "Postman Stories"  
"Circles and Parabolas" (grade 6; 41 minutes)  
"Second Lesson"  
"Graphing an Ellipse" (grade 7; 21 minutes)  
Chapter 10 "First Lesson"  
"Second Lesson"  
Chapters 11-14 (Some films now in preparation.)  
Chapter 15 "Introduction to Truth Tables and Inference Schemes" (grade 7; 40 minutes)  
Chapter 16 "Clues" (grade 6; 20 minutes)  
Chapter 17 "Average and Variance" (grade 6; 40 minutes)  
Chapter 18 "Second Lesson"  
"Introduction to Identities" (grades 3-7; 19 minutes)  
"Accumulating a List of Identities" (grade 6; 21 minutes)

\*The Tic Tac Toe game played in the film "A Lesson with Second Graders" uses the unsatisfactory rule that "5 marks in a straight line constitutes a victory"; as presented in Chapter 2 in the present book, the game uses a more satisfactory rule: using a 5-by-5 board, the rule is that "4 marks in an uninterrupted straight line constitutes a victory."

- Chapter 19 Tape recording D-1 (grade 5)  
 "Making up Identities" (grade 5; 33 minutes)
- Chapter 20 "Making up Identities"  
 "Axioms and Theorems" (grade 6; 1 hour)
- Chapter 22 "Second Lesson"
- Chapter 23 (Various films of ninth-graders at Nerinx High School, Webster Groves, Missouri; for information write to the Madison Project.)
- Chapter 24 (Some films by David Page may be available from Educational Services, Incorporated, Watertown, Massachusetts 02172.)
- Chapter 25 "Guessing Functions" (grade 7; 22 minutes)
- Chapter 27 "A Week of Mathematical Exploration—Parts 2 through 5" (grades 4-5; Tuesday, 33 minutes; Wednesday, 35 minutes; Thursday, 29 minutes; Friday, 36 minutes)  
 "The Study of Functions—Linear, Quadratic, and Exponential" (grades 4-6)
- Chapters 33-34 "Derivation of the Quadratic Formula—First Beginnings" (grades 5-6; 20 minutes)  
 "Derivation of the Quadratic Formula—Final Summary" (grade 7; 20 minutes)  
 "Quadratic Equations" (grade 9; 49 minutes)
- Chapters 37-39 "Matrices" (grades 5-6; 35 minutes)  
 "Solving Equations with Matrices" (grade 6; 36 minutes)  
 "Complex Numbers via Matrices" (grade 7; 33 minutes)  
 "Introduction to the Complex Plane" (grade 9; 54 minutes)
- Chapters 44-45 "Complex Numbers via Matrices"  
 "Solving Equations with Matrices"
- Other films of interest:  
 "Graphing a Parabola" (grade 6; 22 minutes)  
 "Graphs and Truth Sets" (grade 2; 30 minutes)  
 "Weights and Springs" (grade 6; 30 minutes)

## Appendix C

# Some Special Symbols and Concepts Used in Explorations

This brief review of concepts and notations is intended to help you recall or locate ideas in *Explorations*. A "brief" use of language necessarily opens the door to considerable ambiguity, and may suggest the unhappy device of *telling* the students instead of allowing them to learn. If you use this appendix at all, please be careful to *try* to avoid these errors.

A more suitable way to learn these ideas has been presented—we hope—in the main body of this book.

**Variables.** Any of the symbols  $\square$ ,  $\triangle$ ,  $\nabla$ ,  $\square$ ,  $n$ ,  $A$ ,  $B$ ,  $x$ ,  $y$ ,  $\alpha$ ,  $\beta$  may be used to indicate the places in a formula where we may insert numbers or algebraic expressions. When used in this way, any of these symbols would be said to indicate a *variable*.

Examples:  $3 + \square = 5$

Inserting 7 into the  $\square$  yields the false statement  $3 + 7 = 5$ .

Inserting 2 into the  $\square$  yields the true statement  $3 + 2 = 5$ .

$$\square = \square$$

Inserting  $A + B$  into the  $\square$  yields  $A + B = A + B$ .

The use of variables is governed by several important conventions, particularly the rule for substituting, the idea of replacement set, and the notation UV, which are explained below.

**Open sentence.** A sentence that involves a *variable* is called an *open sentence*.

Examples:  $3 + \square = 5$

$\square + \square = 2 \times \square$  (Note that in this example *there is only one variable*—namely,  $\square$ —although this variable occurs three times.)

$\square + \triangle = \triangle + \square$  (This example involves *two variables*— $\square$  and  $\triangle$ —each of which occurs twice.)

$x^2 - \alpha x + \beta = \gamma$  (Involves four variables— $x$ ,  $\alpha$ ,  $\beta$ , and  $\gamma$ ; in ordinary use, in this form,  $\alpha$ ,  $\beta$ , and  $\gamma$  would be "parameters" or "constants," and  $x$  would be an "unknown!")

**Replacement set.** For each variable—for example,  $\square$ —we must agree upon a definite *replacement set*, that is, upon a set of mathematical objects (or names of mathematical objects) that may be written into the "formula" (or "sentence") at the points indicated by that variable.

Example: In  $\square \times 0 = 0$  we might agree that we would write names of *positive integers* in the place indicated by  $\square$ . With this agreement, the *replacement set* for the variable  $\square$  would be

$$R_{\square} = \{^*1, ^*2, ^*3, ^*4, \dots\},$$

and the replacements

$$^*3 \times 0 = 0, ^*25 \times 0 = 0, ^*1066 \times 0 = 0$$

would be "legal," whereas the replacements

$$\frac{1}{2} \times 0 = 0, ^{-}1 \times 0 = 0$$

would not be.

**Rule for substituting (one variable).** If the same letter (or the same "shape," such as  $\square$ ) occurs several times in the same open sentence, you may legally use *any* element of the replacement set as a replacement for the first occurrence but you must then use *this same number* as a replacement for all other occurrences of  $\square$  in that open sentence.

Example: If  $R_{\square} = \{0, 1, 2, 3, 4, \dots\}$  and the open sentence is  $(\square \times \square) - (5 \times \square) + 6 = 0$ , you may use 9—or any other element of  $R_{\square}$ —as a replacement for the first occurrence of  $\square$ ,

$$(\boxed{9} \times \square) - (5 \times \square) + 6 = 0,$$

provided that you use this *same number* as a replacement for all other occurrences of  $\square$ ,

$$(\boxed{9} \times \boxed{9}) - (5 \times \boxed{9}) + 6 = 0.$$



**Rule for substituting (several variables).** If several variables occur in the same open sentence, the *rule for substituting* applies to each one independently. In particular, it is legal to put the same number in different shapes, although it would not be legal to put different numbers into the same shape.

Examples: Legal substitutions

$$\begin{aligned} \boxed{1} + \boxed{1} + \triangle 1 + \triangle 1 &= 100 \\ \boxed{2} + \boxed{2} + \triangle 3 + \triangle 3 &= 100 \\ \boxed{20} + \boxed{20} + \triangle 30 + \triangle 30 &= 100 \\ \boxed{25} + \boxed{25} + \triangle 25 + \triangle 25 &= 100 \end{aligned}$$

Notice that the "legality" of a replacement for a variable does not depend upon whether the resulting statement is true or false.

Examples: Illegal substitutions

$$\begin{aligned} \boxed{10} + \boxed{20} + \triangle 30 + \triangle 40 &= 100 \\ \boxed{1} + \boxed{2} + \triangle 3 + \triangle 3 &= 100 \end{aligned}$$

**Open names as replacements for variables.** If 3 is an element of  $R_{\square}$ , then using 3 as a replacement for  $\square$  is legal.

Example:  $\square + 0 = \square$   
 $3 \rightarrow \square$   
 $3 + 0 = 3$

We call this a "numerical replacement" for the variable  $\square$ . It is also legal to take an *open name* (which itself involves variables) and use this as a replacement for a variable.

Example:  $\square + 0 = \square$   
 $A + B \rightarrow \square$   
 $(A + B) + 0 = A + B$

Of course,  $A + B$  must then be used to name an element of  $R_{\square}$  (but this is not usually something you will need to worry about). For example, if  $R_{\square} = \{1, 2, 3, 4, \dots\}$ , then we could agree to take  $R_A = \{1, 2, 3, 4, \dots\}$  and  $R_B = \{1, 2, 3, \dots\}$ , since this set is closed under addition, and—no matter which element of  $R_A$  and  $R_B$  we choose—the expression  $A + B$  will turn out to be the name for some element of  $R_{\square}$ .

**UV.** The notation UV (use of variables) is employed whenever we want to show how we have carried out a replacement for some variable. This is particularly the case in writing derivations. Our usual method for writing UV is  $UV: 3 \rightarrow \square$ .

Example:

A piece of a derivation:	Reason:
(i) $\square + 0 = \square$	ALZ
(ii) $(A + B) + 0 = A + B$	UV: $A + B \rightarrow \square$ , in line (i)

**Truth set.** In most open sentences it is possible to carry out a replacement for the variable (or variables) so as to get a *true statement* as a result, or else to use a different replacement that will yield a *false statement* as a result. Those elements of the replacement set which yield *true statements* are said to form, collectively, the *truth set* for the open sentence.

Example: If  $3 < \square < 6$ ,  $R_{\square} = \{1, 2, 3, 4, 5, 6, 7, \dots\}$ , then the *truth set*  $T$  is given by  $T = \{4, 5\}$ . Similarly, one defines the *false set*  $F$ , so that, in the example above,  $F = \{1, 2, 3, 6, 7, 8, 9, \dots\}$ .

**Set.** Roughly, any collection of things is a *set*. Thus, one can speak of "the set of positive integers," or "the set of 2-by-2 matrices," and so forth. Now, actually, matters are more complicated than this, as the work of Bertrand Russell, Kurt Gödel, and others has shown. Considerable nicety and precision is required in defining sets if we want to be able to deal with them in a precise and abstract way. This probably need not concern us here, but we should not be under the illusion that the "sets" we are talking about are actually the same things as the "super-refined" sets that are discussed by Russell and other modern logicians. [Anyone interested in expending the large amount of time and effort that is required to study modern logic may wish to consult Appendix A, Curry (89)].

That precise reasoning with sets involves pitfalls and hazards should be suggested by the following, which is attributable to Bertrand Russell: Some sets are elements (or "members") of other sets—i.e., some "collections" are in fact "collections of collections." Thus, if John and I both bring our stamp collections over to Billy's house, then we have, assembled in one spot, a collection of three collections (or a "set of three sets," a "set whose elements are themselves sets," and so on). In some sets, on the other hand, the elements are not themselves sets—for example, in the set  $\{2, 3\}$  the "elements" or "members" are the *numbers* 2 and 3.

Can a set ever be an element of itself? The answer is that, in our present unsophisticated use of language, a set surely may be an element of itself. For example, the set of "all the sets in the world" is *itself* a set, and so it must be a member of the set of all the sets in the world—i.e., it is a member of itself.

All right, let us now divide all the sets in the world into two categories. We shall put them in Category A if they are a member of themselves, and into Category B if they are not. Now, Category B is, itself, a collection—that is to say,

it is (in our rough language) a set. Is Category B a member of itself, or not? Well, *if it is* a member of itself, then it *should* have been put into Category A, and *not* into Category B. Hence, if Category B is a member of itself, *it should not be*.

On the other hand, if Category B is *not* a member of itself, then it should have been assigned to Category B, and hence *would be* a member of itself. That is, if Category B is *not* a member of itself, *then it should be*.

Into which category, then, ought you to put Category B?

I mention this kind of difficulty only to temper the fads of the times. If we seize upon "sets" as the essential ingredient of "new mathematics," we are opening Pandora's box. Instead, I suggest we use the word only in a commonplace "colloquial" sense, to mean "a collection," and I suggest we use it *only when it is natural to be speaking of a collection of things*—as, for example, when we wish to speak of the collection of numbers that will make

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

become a *true* statement when we write one of the numbers in the  $\square$ , using UV.

A common notation for sets is  $\{2, 3\}$ ,  $\{1, 2, 3, 4, \dots\}$ , and so forth.

**The terminal three dots.** When a list "trails off" and ends in three dots, we are thereby trying to indicate that the list "truly" goes on and on forever, and never stops.

Example: The "counting numbers" are 1, 2, 3, ... (or, depending upon which book you are reading, perhaps they are 0, 1, 2, 3, ...).

**Subset.** The set  $A$  is called a *subset* of set  $B$  if every element of  $A$  is also an element of  $B$ . Thus, if  $A = \{2, 3, 7\}$  and  $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ , then  $A$  is a subset of  $B$ , and we write  $A \subset B$ .

**Element versus subset.** The *individual* "things" that make up a set we call its "elements." This can—hopefully—be made clearer by some examples.

Examples:  $A = \{1, 2, 3, 4\}$

Then 1 is an element of  $A$ , 2 is an element of  $A$ , 3 is an element of  $A$ , 4 is an element of  $A$ .

$P = \{\{1, 2\}, \{2, 3\}\}$

This requires caution! The number 1 *is not* an element of  $P$ . The number 2 *is not* an element of  $P$ . The number 3 *is not* an element of  $P$ . In fact, the set  $P$  has two elements: The set  $\{1, 2\}$  is an element of  $P$ . The set  $\{2, 3\}$  is an element of  $P$ .

In case it helps any, consider these somewhat analogous examples:

Examples: The Mid-city library may be a member of the National Inter-library Loan Association, and

Webster's dictionary may be in the collection of the Mid-city Library. However, Webster's dictionary is not a member of the National Inter-library Loan Association.

The United States holds membership in the United Nations, and you may be a member of the United States. This does *not* imply that you hold membership in the United Nations.

Notice that *elements* and *subsets* are quite different. For the set  $W = \{2, 4, 6\}$  we have:

Elements: The number 2 is an element of  $W$ . The number 4 is an element of  $W$ . The number 6 is an element of  $W$ .

Subsets: The set  $\{2, 4, 6\}$  is a subset of  $W$ . The set  $\{2, 4\}$  is a subset of  $W$ . The set  $\{2, 6\}$  is a subset of  $W$ . The set  $\{4, 6\}$  is a subset of  $W$ . The set  $\{2\}$  is a subset of  $W$  (but  $\{2\} \neq 2$ ). The set  $\{4\}$  is a subset of  $W$ . The set  $\{6\}$  is a subset of  $W$ . The set  $\phi$  is a subset of  $W$  (where " $\phi$ " denotes the "empty set").

The symbol  $\in$  is used to mean "is an element of"; thus  $2 \in W$ . The symbol  $\subset$  is used to mean "is a subset of"; thus  $\{2, 4\} \subset \{2, 4, 6\}$ ,  $\{6\} \subset W$ , and so on.

**The empty set.** The symbol  $\phi$  denotes a set which is empty. If you think carefully about the truth table for the statement  $P \Rightarrow Q$ , you will see that every statement which begins "if  $x$  is an element of the empty set  $\phi$ , then ..." must be *true*, *no matter how the statement ends*. For this reason, the empty set is always a subset of *any* set  $J$ , no matter what set  $J$  is.

Notice that there is a difference between  $\phi$  and  $\{0\}$ .

**The Cartesian product of two sets,  $A \times B$ .** See Chapter 2.

**Function and functional notation.** A function may be written as  $f(x)$ :

$$\begin{aligned} f(1) &= 3 \\ f(2) &= 5 \\ f(3) &= 7 \\ &\vdots \\ &\vdots \end{aligned}$$

Or it may be written as a "mapping":

$$\begin{aligned} f:1 &\longrightarrow 3 \\ f:2 &\longrightarrow 5 \\ f:3 &\longrightarrow 7 \end{aligned}$$

Notice that the arrow symbol used to indicate a mapping is longer than the arrow symbol used in connection with UV.

**Truth value.** A "statement" may be either *true* or *false*. The assessment "true," when appropriate, is called the "truth value" of the statement; similarly, the assessment "false," if appropriate, is called the "truth value" of the statement.

Examples:	Statement	Truth Value
	St. Louis is a city.	T
	Missouri is a large city in eastern Massachusetts.	F

We usually write truth values merely as T or as F.

When we make up an abstract system where these are the only truth values allowed, we say we have made up a "two-valued" logic. When (as children often choose to do, and adult mathematicians sometimes do also) we make up a system where more than two truth values are allowed (for example, "true," "false," and "sort of"), we say we have made up a "many-valued logic."

**PN.** Standing for the "principle of names," PN means that if, in any statement, you "erase" a name for something and put in *another name for that same thing*, you will *not* change the truth value of the statement. If it was false before, it still is. If it was true before, it still is.

Example: Suppose

Eileen = Miss Godfrey.

Then, in the statement

Eileen is ten feet tall,

we could erase "Eileen," and put in "Miss Godfrey," and get

Miss Godfrey is ten feet tall.

Presumably, both statements are false.

**Opposite or additive inverse.** We have occasionally used the word *opposite*, as is common in recent books. However, do not let yourself be confused by the *everyday* meaning of the word, which is quite different from its *mathematical meaning*. In its *mathematical meaning*, *opposite* (or perhaps better, *additive inverse*) of a number *A* means: *the number you must add to A in order to get zero*.

Example:  $'3 + \underline{\quad} = 0$

↑  
"3 is the additive inverse of '3.